EA3 Quiz 2
Section 21, Spring 2023

Name: ________________________________

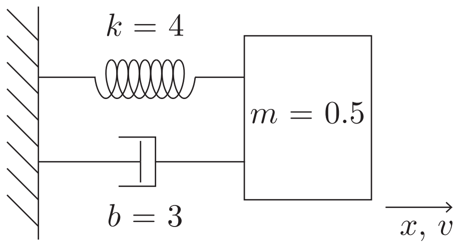
You are allowed to use only pens, pencils, and erasers. No electronics, other papers, etc.

Make sure to show all your work, and make sure your final answer is clear (for example, you can circle it). Full credit, or partial credit if your final answer is wrong, will only be given if your thought process is clear. If you think any question does not give you enough information to give an answer (i.e., there is a mistake on the test), then clearly write the extra assumptions you had to make to answer the question, and answer the question using those assumptions.

If you get a square root in your answer, like $\sqrt{12}$, it is preferred that you write it in its reduced form, $2\sqrt{3}$. Leaving it as $\sqrt{12}$ is also OK, but not preferred. DO NOT try to express it as an approximate decimal value, e.g., $\sqrt{12} = 3.46$ unless it has a simple exact decimal value.

Use the backs of the pages if you need more room for your work.

1. Consider the mass-spring-damper in the figure below. There is no gravity. For all answers below, use the numerical values for the spring constant, damping constant, and mass. (Don't write k, b, and m in any answers).



(a) (2 pts) If we are using meters (m) for distance, kilograms (kg) for mass, and seconds (s) for time, what are the units of the spring constant k and the damping constant b? Use only combinations of m, kg, and/or s in your answers; no other units. (In the rest of the questions in this problem, you do not have to give units unless requested.)

(b) (2 pts) For the state vector $y = [x, v]^{\mathsf{T}}$, write the matrix A for the state equations $\dot{y} = Ay$.

(c) (2 pts) Write the corresponding second-order linear homogeneous differential equation in terms of $\ddot{x}$, $\dot{x}$, and x.

(d) (2 pts) Write the corresponding characteristic equation as a polynomial of s. (Also called r in the webtext.)

(e) (3 pts) Find the numerical solutions s_1 and s_2 of the characteristic equation. Is the system overdamped, underdamped, critically damped, or none of the above?

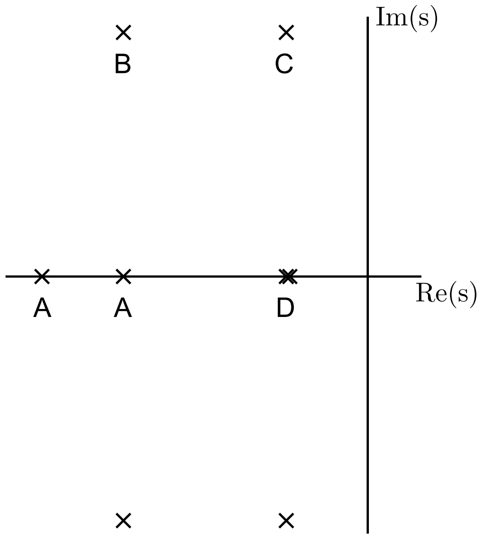
(f) (2 pts) Give the form of the general solution $x(t)$ using your numerical solutions for s_1 and s_2.

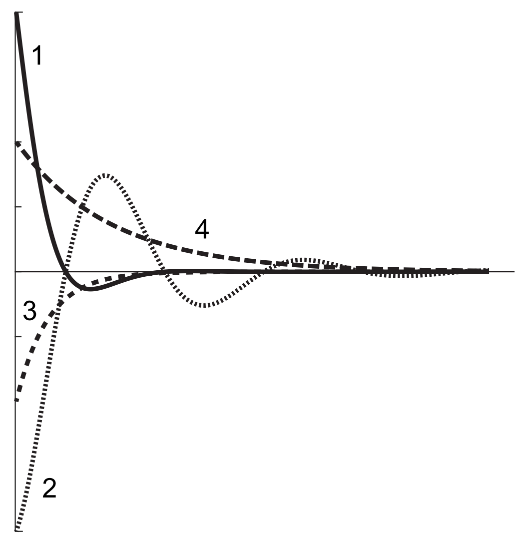
(g) (2 pts) If the system as drawn in the figure is not critically damped, what damping constant b would you choose (leaving the mass and spring unchanged) to make the system critically damped? (If the original system is critically damped, simply answer "no change to the damping constant.")

(h) (2 pts) You decide instead to choose a damping constant b to make the damped natural frequency 2 rad/s. (Leave the mass and spring constant at their original values.) What damping constant do you choose?

(i) (2 pts) For the damping constant you chose in your previous answer, the general solution is a sinusoid multiplied by a decaying exponential of the form e^{ct}. What is c? **Give units for your answer.**

2. The figure on the left below shows the solutions to the characteristic equations (the "roots") of the second-order linear homogeneous differential equations of four different systems: A, B, C, and D. (The two roots for D are at the same location.) On the right are particular solutions for systems A–D given particular initial conditions.





(a) (4 pts) The right figure shows one particular solution (1–4) for each system A, B, C, and D. For each of the four systems A–D, indicate which solution 1–4 corresponds to that system. Explain your answers in a sentence or less. (Understandable explanations must be given to receive credit.)

A.

B.

C.

D.

(b) (4 pts) Each of the four systems A–D is either overdamped, underdamped, critically damped, or none of the above. For each, indicate the system type and explain your answer in a sentence or less. (Understandable explanations must be given to receive credit.)

A.

B.

C.

D.

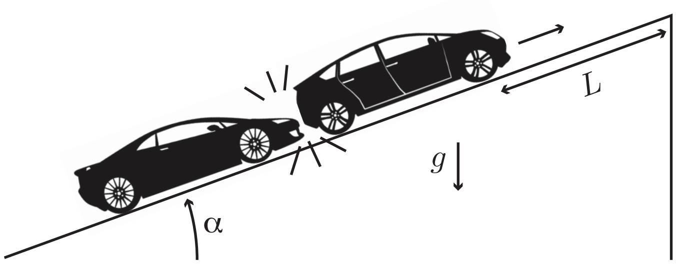
3. Two solutions to a second-order differential equation are $x_1(t) = e^{j\omega t}$ and $x_2(t) = e^{-j\omega t}$, and the general solution can be written $Ax_1(t) + Bx_2(t)$, where A and B are constants.

   (a) (2 pts) Find A and B so that $Ax_1(t) + Bx_2(t) = \cos(\omega t)$.

   (b) (2 pts) Find A and B so that $Ax_1(t) + Bx_2(t) = \sin(\omega t)$.

   (c) (2 pts) The general solution can also be written as $C\cos(\omega t) + D\sin(\omega t)$. If the initial conditions are $x(0) = 1$ and $\dot{x}(0) = -3$, what are C and D for the particular solution?

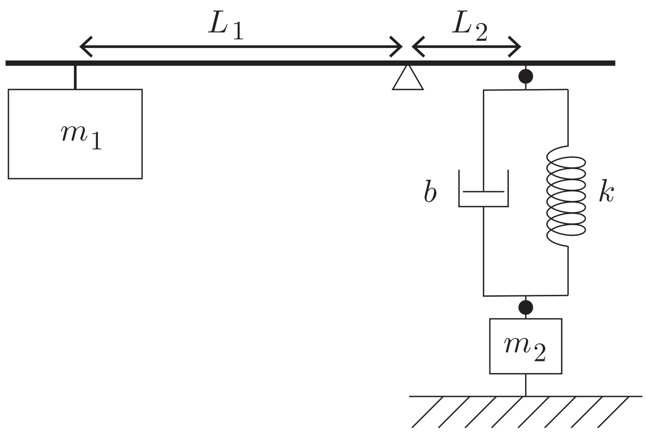
4. Two cars are driving up a mountain road at a slope α when the road suddenly becomes icy. The back car, mass m_1 and velocity v_1, rear ends the front car, mass m_2 and velocity v_2, in a perfectly plastic (inelastic) impact.



   (a) (2 pts) What is the velocity v of the cars immediately after impact?

   (b) (2 pts) After impact, the wheels lock and the cars slide together over the icy road with a friction coefficient of μ. Gravity is g. The distance from the front wheels of the front car to the cliff is L. What is the minimum distance L to make sure the front of the front car does not dangle over the cliff (or worse yet, plummet) before coming to a stop? Your answer should be in terms of v from the previous part and anything else you need.

5. (6 pts) Consider the mass-spring-damper-lever system below. There is no gravity. When the system is at equilibrium (no motion), the extension of the spring is $x = 0$, and $x > 0$ means the spring is extended. Write the second-order equation of motion of this system in terms of $\ddot{x}$, $\dot{x}$, x, m_1, m_2, b, k, L_1, and/or L_2.



EA3 Quiz 2
Section 21, Spring 2023

Name: Solutions

You are allowed to use only pens, pencils, and erasers. No electronics, other papers, etc.

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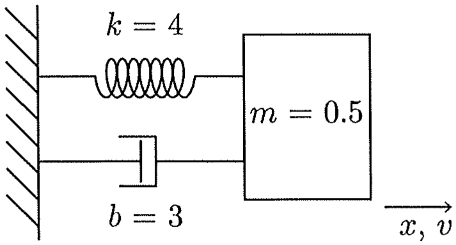
Use the backs of the pages if you need more room for your work.

1. Consider the mass-spring-damper in the figure below. There is no gravity. For all answers below, use the numerical values for the spring constant, damping constant, and mass. (Don't write k, b, and m in any answers).



(a) (2 pts) If we are using meters (m) for distance, kilograms (kg) for mass, and seconds (s) for time, what are the units of the spring constant k and the damping constant b? Use only combinations of m, kg, and/or s in your answers; no other units. (In the rest of the questions in this problem, you do not have to give units unless requested.)

kx has units $N = \frac{kg\, m}{s^2}$, so k has units $\frac{kg}{s^2}$.

bv has units N, so b is $\frac{Ns}{m}$ or $\frac{kg\, m}{s^2}\frac{s}{m} = \frac{kg}{s}$

(b) (2 pts) For the state vector $y = [x, v]^\intercal$, write the matrix A for the state equations $\dot{y} = Ay$.

force balance: $m\dot{v} = -kx - bv$

$$0.5\dot{v} = -4x - 3v$$
$$\dot{v} = -8x - 6v$$

and $\dot{x} = v$ So

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \end{bmatrix}$$

(c) (2 pts) Write the corresponding second-order linear homogeneous differential equation in terms of $\ddot{x}$, $\dot{x}$, and x.

$$\dot{v} = -8x - 6v \Rightarrow \ddot{x} = -8x - 6\dot{x} \Rightarrow \ddot{x} + 6\dot{x} + 8x = 0$$

(d) (2 pts) Write the corresponding characteristic equation as a polynomial of s. (Also called r in the webtext.)

$$s^2 + 6s + 8 = 0$$

(replace $\ddot{x}$ by s^2, $\dot{x}$ by s, x by $1 = s^0$)

(e) (3 pts) Find the numerical solutions s_1 and s_2 of the characteristic equation. Is the system overdamped, underdamped, critically damped, or none of the above?

$$s_{1,2} = \frac{-6 \pm \sqrt{6^2 - 4\cdot 8}}{2} = \frac{-3 \pm \sqrt{4}}{2} = -4, -2$$

$\boxed{\text{2 real roots, overdamped} \quad s_1 = -4 \quad s_2 = -2}$

(f) (2 pts) Give the form of the general solution $x(t)$ using your numerical solutions for s_1 and s_2.

$$x(t) = Ae^{-4t} + Be^{-2t}$$

(g) (2 pts) If the system as drawn in the figure is not critically damped, what damping constant b would you choose (leaving the mass and spring unchanged) to make the system critically damped? (If the original system is critically damped, simply answer "no change to the damping constant.")

$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 6s + 8$ (original) So $\omega_n = \sqrt{8} = 2\sqrt{2}$

For critical damping, $\zeta = 1$, so we need $s^2 + (2\cdot 1\cdot 2\sqrt{2})s + 8 = 0$

this is $\frac{b}{m} = \frac{b}{0.5}$ $\quad s^2 + \underbrace{4\sqrt{2}}\, s + 8 = 0$

so $\boxed{b = 2\sqrt{2}}$

(h) (2 pts) You decide instead to choose a damping constant b to make the damped natural frequency 2 rad/s. (Leave the mass and spring constant at their original values.) What damping constant do you choose?

$\omega_d = \omega_n\sqrt{1-\zeta^2}$

$2 = 2\sqrt{2}\sqrt{1-\zeta^2}$

$4 = 4\cdot 2(1-\zeta^2)$

$4 = 8 - 8\zeta^2$

$8\zeta^2 = 4$

$\zeta = \sqrt{\frac{1}{2}}$

$\frac{b}{m} = 2\sqrt{\frac{1}{2}}\; 2\sqrt{2} = 2\zeta\omega_n$

$\frac{b}{m} = 4 \xrightarrow{\hspace{1cm}} 0.5$

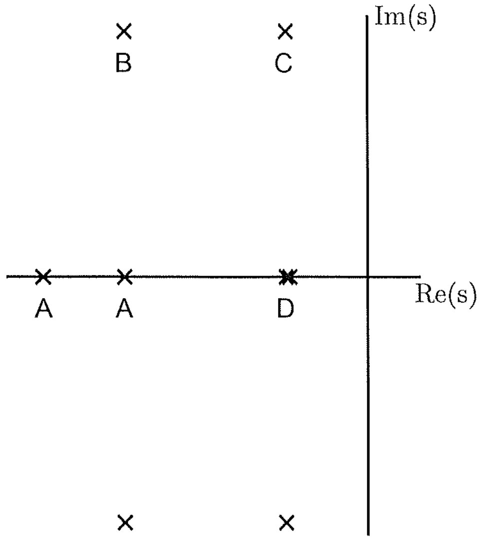
so $\boxed{b = 2}$

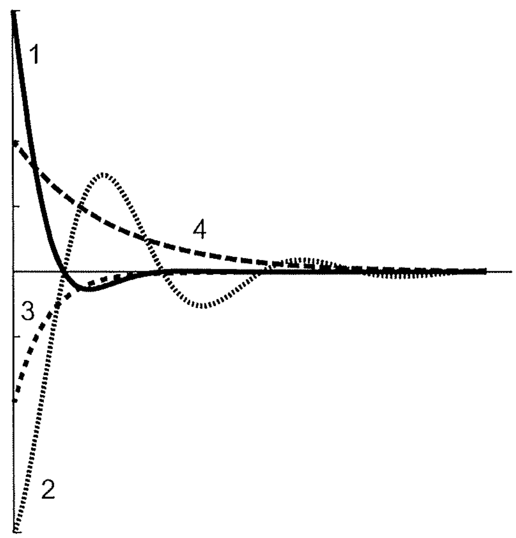
(i) (2 pts) For the damping constant you chose in your previous answer, the general solution is a sinusoid multiplied by a decaying exponential of the form e^{ct}. What is c? **Give units for your answer.**

$x(t) = e^{-\zeta\omega_n t}$ (sinusoid)

$c = -\zeta\omega_n = -\frac{1}{\sqrt{2}}\, 2\sqrt{2} = -2\, s^{-1}$ or $-2\,\frac{1}{s}$

2. The figure on the left below shows the solutions to the characteristic equations (the "roots") of the second-order linear homogeneous differential equations of four different systems: A, B, C, and D. (The two roots for D are at the same location.) On the right are particular solutions for systems A–D given particular initial conditions.





(a) (4 pts) The right figure shows one particular solution (1–4) for each system A, B, C, and D. For each of the four systems A–D, indicate which solution 1–4 corresponds to that system. Explain your answers in a sentence or less. (Understandable explanations must be given to receive credit.)

A. 3. Overdamped, so no oscillation. Fast decay, since roots ~~h~~ave large negative real numbers.

B. 1. Underdamped, so oscillation, but faster decay than C.

C. 2. Underdamped, slower decay than B.

D. 4. Critically damped, slow decay, no oscillation.

(b) (4 pts) Each of the four systems A–D is either overdamped, underdamped, critically damped, or none of the above. For each, indicate the system type and explain your answer in a sentence or less. (Understandable explanations must be given to receive credit.)

A. Overdamped 2 different real roots

B. Underdamped complex conjugate roots

C. Underdamped Complex conjugate roots

D. Critically damped 2 identical real roots

3. Two solutions to a second-order differential equation are $x_1(t) = e^{j\omega t}$ and $x_2(t) = e^{-j\omega t}$, and the general solution can be written $Ax_1(t) + Bx_2(t)$, where A and B are constants.

(a) (2 pts) Find A and B so that $Ax_1(t) + Bx_2(t) = \cos(\omega t)$.

$A(\cos\omega t + j\sin\omega t) + B(\cos\omega t - j\sin\omega t) = \cos\omega t$

$(A+B)\cos\omega t + (A-B)j\sin\omega t = \cos\omega t$ $\quad A-B=0$, $A+B=1$ so $A=\frac{1}{2}, B=\frac{1}{2}$

(b) (2 pts) Find A and B so that $Ax_1(t) + Bx_2(t) = \sin(\omega t)$.

$(A+B)\cos\omega t + (A-B)j\sin\omega t = \sin\omega t$

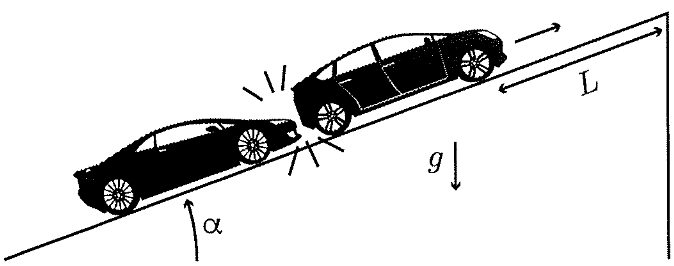
$A+B=0 \quad A-B=-j \Rightarrow A = -\frac{j}{2}, \; B = \frac{j}{2}$

(c) (2 pts) The general solution can also be written as $C\cos(\omega t) + D\sin(\omega t)$. If the initial conditions are $x(0) = 1$ and $\dot{x}(0) = -3$, what are C and D for the particular solution?

$x(0) = 1 = C\cos 0 + D\sin 0 \Rightarrow C = 1$

$\dot{x}(0) = -3 = -\omega C\sin 0 + \omega D\cos 0 \Rightarrow D = \frac{-3}{\omega}$

4. Two cars are driving up a mountain road at a slope α when the road suddenly becomes icy. The back car, mass m_1 and velocity v_1, rear ends the front car, mass m_2 and velocity v_2, in a perfectly plastic (inelastic) impact.



(a) (2 pts) What is the velocity v of the cars immediately after impact?

Cons. momentum: $(m_1+m_2)V = m_1V_1 + m_2V_2 \Rightarrow V = \frac{m_1V_1 + m_2V_2}{m_1+m_2}$

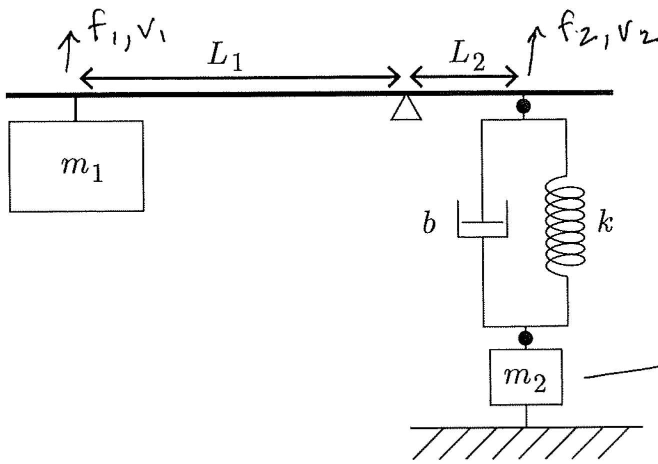
(b) (2 pts) After impact, the wheels lock and the cars slide together over the icy road with a friction coefficient of μ. Gravity is g. The distance from the front wheels of the front car to the cliff is L. What is the minimum distance L to make sure the front of the front car does not dangle over the cliff (or worse yet, plummet) before coming to a stop? Your answer should be in terms of v from the previous part and anything else you need.

initial energy $= \frac{1}{2}(m_1+m_2)v^2$

when coming to a stop, PE gained + lost due to friction is $mgL\sin\alpha + \mu mgL\cos\alpha$, where $m = m_1+m_2$.

$\frac{1}{2}v^2 = gL\sin\alpha + \mu gL\cos\alpha \Rightarrow L = \frac{v^2}{2g(\sin\alpha + \mu\cos\alpha)}$

5. (6 pts) Consider the mass-spring-damper-lever system below. There is no gravity. When the system is at equilibrium (no motion), the extension of the spring is $x = 0$, and $x > 0$ means the spring is extended. Write the second-order equation of motion of this system in terms of $\ddot{x}$, $\dot{x}$, x, m_1, m_2, b, k, L_1, and/or L_2.



→ Ignore this mass!
It is attached to ground.

$f_2 = -kx - bv = -kx - b\dot{x}$: force balance + constitutive

lever: $f_1 L_1 = f_2 L_2$, $\frac{v_1}{L_1} = -\frac{v_2}{L_2}$

mass force balance: $m_1 \dot{v}_1 = -f_1$ ← get this in terms of f_2, v_2 using lever

$$-m_1 \dot{v}_2 \frac{L_1}{L_2} = -f_2 \frac{L_2}{L_1}$$ → note $v_2 = \dot{x}$, $\dot{v}_2 = \ddot{x}$

$$-m_1 \frac{L_1}{L_2} \ddot{x} = -(-kx - b\dot{x}) \frac{L_2}{L_1}$$

$$m_1 \left(\frac{L_1}{L_2}\right)^2 \ddot{x} = -kx - b\dot{x}$$

$$m_1 \left(\frac{L_1}{L_2}\right)^2 \ddot{x} + b\dot{x} + kx = 0$$