EA3 Quiz 2
Prof. Lynch
Spring 2024

Name: ____________________________________

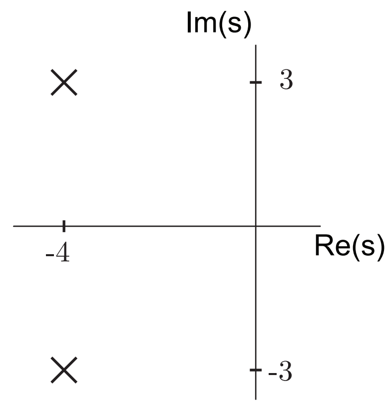
You are allowed to use only pens, pencils, and erasers. No electronics, other papers, etc.

Make sure to show all your work, and make sure your final answer is clear (for example, you can circle it). **Use the back of the previous page if you need more space, for your work or your final answer.** Full credit, or partial credit if your final answer is wrong, will only be given if your thought process is clear. If you think any question does not give you enough information to give an answer (i.e., there is a mistake on the test), then clearly write the extra assumptions you had to make to answer the question, and answer the question using those assumptions.

No significant calculations are needed, so you don't need a calculator. If you get an answer like $11/32$ or $\sqrt{5}$, just leave it like that. On the other hand, we would appreciate simple calculations, like reducing $6/2$ to 3 or $\sqrt{12}$ to $2\sqrt{3}$.

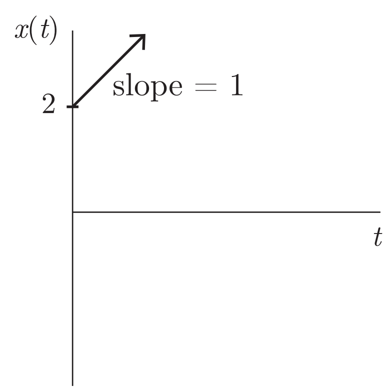
The "standard" second-order characteristic equation is $s^2+2\zeta\omega_n s+\omega_n^2=0$. The roots of a quadratic equation $as^2+bs+c=0$ are $s_{1,2}=(-b\pm\sqrt{b^2-4ac})/(2a)$.

1. You are given a second-order, linear, constant coefficient, homogeneous differential equation in terms of x, $\dot{x}$, and $\ddot{x}$.
   (a) (2 pts) The roots of the corresponding characteristic equation are shown in the complex plane below. Is the system overdamped, underdamped, critically damped, or none of the above? Explain.



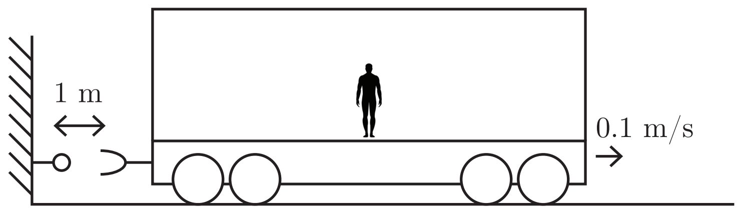
   (b) (2 pts) Give the general solution $x(t)$ of the differential equation. Use numerical (not variable/symbolic) values wherever possible.

   (c) (2 pts) The plot below shows the beginning of a particular solution $x(t)$ at time $t = 0$, including the tangent of the response at $t = 0$. Give the equation for the particular solution $x(t)$. (No unknowns/variables/symbols.)

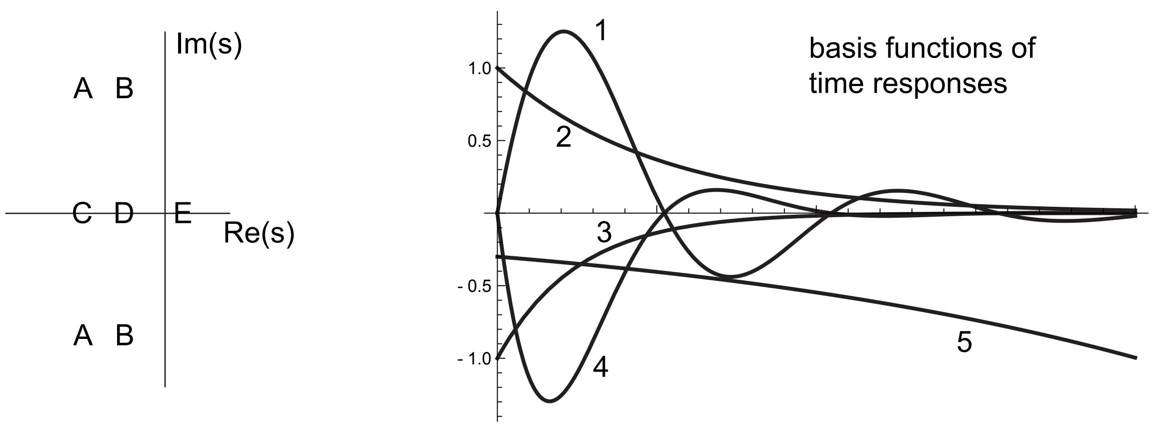


2. (2 pts) Given the equality $a + bj = (c + dj)/(x + yj)$, solve for a and b in terms of c, d, x, and y.

3. (4 pts) A man is standing inside a closed train car which rolls without friction on a track. The car weighs 1000 kg and is rolling to the right relative to the track at 0.1 m/s. The man weighs 100 kg and is capable of running at a top speed of 5 m/s relative to the train car. He must stay inside the car, only running on the floor of the car. The man wants the car to move to the left by 1 m so it couples (attaches) to the wall. Is it possible for the man to make the car couple to the wall? Impossible? Or is there not enough information to answer definitively? Justify your answer mathematically. If there is not enough information, explain what else you would need to know to answer "possible" or "impossible." (Use the back of the previous page if needed.)



4. (5 pts) The left side of the figure below shows the locations of the roots in the complex plane of two second-order systems (AA and BB) and three first-order systems (C, D, and E). The right side of the figure shows five of a total of seven basis functions of the time responses for the five systems. For each of the five time responses 1–5, indicate the system (AA, BB, C, D, or E) for which it is a possible time response and explain your answer in a sentence or less to get full credit. (The same system may be the answer for more than one time response.)



1:

2:

3:

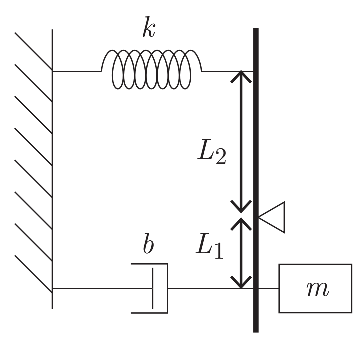
4:

5:

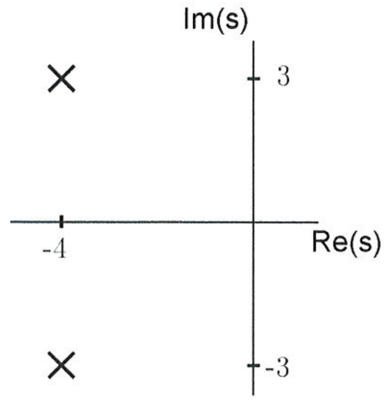
5. The dynamics of an unstable uncontrolled system are $\ddot{x} - 3\dot{x} + 9x = 0$. You want to stabilize it using derivative feedback control, which yields the new system dynamics $\ddot{x} + (K_D - 3)\dot{x} + 9x = 0$, where K_D is a control gain you can choose.
   (a) (2 pts) What K_D do you choose to achieve critical damping?

   (b) (2 pts) What is the time constant τ of the exponential decay of the response $x(t)$ of the new critically damped system?

6. (6 pts) Let x be the extension of the spring in the lever-mass-spring-damper system shown below. The second-order differential equation for the system can be written $\ddot{x} + p\dot{x} + qx = 0$. Find p and q in terms of the constants in the figure.



1. You are given a second-order, linear, constant coefficient, homogeneous differential equation in terms of x, $\dot{x}$, and $\ddot{x}$.
   (a) (2 pts) The roots of the corresponding characteristic equation are shown in the complex plane below. Is the system overdamped, underdamped, critically damped, or none of the above? Explain.



Underdamped. Roots are complex.

In unnecessary detail:

$(s+4-3j)(s+4+3j) =$

$s^2 + 4s - \cancel{3js} + 4s + 16 - \cancel{12j} + \cancel{3js} + \cancel{12j} + 9$

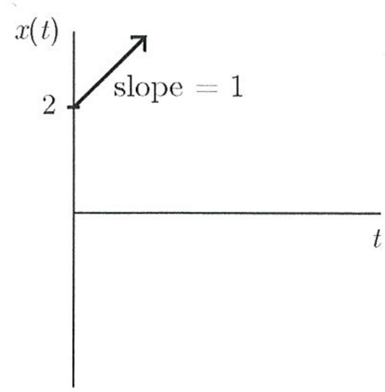
$s^2 + 8s + 25 \qquad \omega_n^2 = 25 \rightarrow \omega_n = 5$

$2\zeta\omega_n = 8 \rightarrow \zeta = \frac{4}{5}$, less than 1.

   (b) (2 pts) Give the general solution $x(t)$ of the differential equation. Use numerical (not variable/symbolic) values wherever possible.

$x(t) = e^{-4t}(A\cos 3t + B\sin 3t)$

   (c) (2 pts) The plot below shows the beginning of a particular solution $x(t)$ at time $t = 0$, including the tangent of the response at $t = 0$. Give the equation for the particular solution $x(t)$. (No unknowns/variables/symbols.)



$x(0) = 2 = A$

$\dot{x} = -4e^{-4t}(A\cos 3t + B\sin 3t)$
$\quad + e^{-4t}(-3A\sin 3t + 3B\cos 3t)$

$\dot{x}(0) = 1 = -4A + 3B$

$1 = -8 + 3B \rightarrow B = 3$

$x(t) = e^{-4t}(2\cos 3t + 3\sin 3t)$

2. (2 pts) Given the equality $a + bj = (c + dj)/(x + yj)$, solve for a and b in terms of c, d, x, and y.

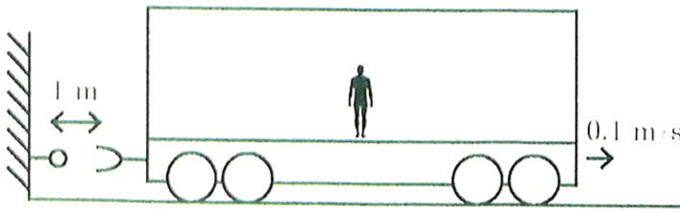
$a + bj = \frac{c + dj}{x + yj} \cdot \frac{x - yj}{x - yj} = \frac{cx - cyj + dxj + dy}{x^2 + y^2}$

$a = \frac{cx + dy}{x^2 + y^2} \qquad b = \frac{dx - cy}{x^2 + y^2}$

3. (4 pts) A man is standing inside a closed train car which rolls without friction on a track. The car weighs 1000 kg and is rolling to the right relative to the track at 0.1 m/s. The man weighs 100 kg and is capable of running at a top speed of 5 m/s relative to the train car. He must stay inside the car, only running on the floor of the car. The man wants the car to move to the left by 1 m so it couples (attaches) to the wall. Is it possible for the man to make the car couple to the wall? Impossible? Or is there not enough information to answer definitively? Justify your answer mathematically. If there is not enough information, explain what else you would need to know to answer "possible" or "impossible." (Use the back of the previous page if needed.)

$\boxed{\text{Not enough info.}}$

Momentum is conserved. Initially, $(1000+100)\,0.1 = 110\ kg\frac{m}{s}$. If man runs 5 m/s to right,



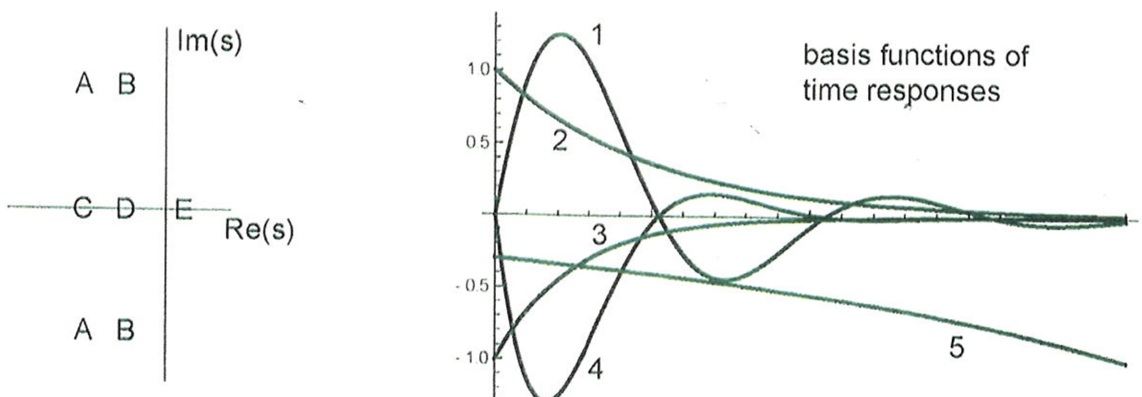
$110 = 1000\, V_{car} + 100(5 + V_{car})$

$-390 = 1100\, V_{car}$

$V_{car} = \frac{-39}{110}$ m/s.

So the man can run fast enough to make the car move to the left, but we need to know how far the man can run in the car, and the man's acceleration.

4. (5 pts) The left side of the figure below shows the locations of the roots in the complex plane of two second-order systems (AA and BB) and three first-order systems (C, D, and E). The right side of the figure shows five of a total of seven basis functions of the time responses for the five systems. For each of the five time responses 1–5, indicate the system (AA, BB, C, D, or E) for which it is a possible time response and explain your answer in a sentence or less to get full credit. (The same system may be the answer for more than one time response.)



1: BB. Oscillation (underdamped 2nd-order) with slow decay.

2: D. No oscillation (1st-order), slow decay.

3: C. 1st-order, fast decay.

4: AA. Oscillation (underdamped 2nd-order), fast decay.

5: E. Growing exponential for real root in right-half plane.

5. The dynamics of an unstable uncontrolled system are $\ddot{x} - 3\dot{x} + 9x = 0$. You want to stabilize it using derivative feedback control, which yields the new system dynamics $\ddot{x} + (K_D - 3)\dot{x} + 9x = 0$, where K_D is a control gain you can choose.

(a) (2 pts) What K_D do you choose to achieve critical damping?

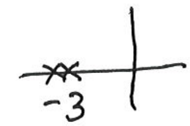
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + (K_D - 3)s + 9 = 0$$

$$\omega_n^2 = 9 \rightarrow \omega_n = 3 \qquad 2\zeta\omega_n = K_D - 3$$

$$2(1)(3) = K_D - 3 \longrightarrow K_D = 9$$

(b) (2 pts) What is the time constant τ of the exponential decay of the response $x(t)$ of the new critically damped system?

$$s^2 + 6s + 9 = (s+3)^2 \qquad \text{roots:}$$

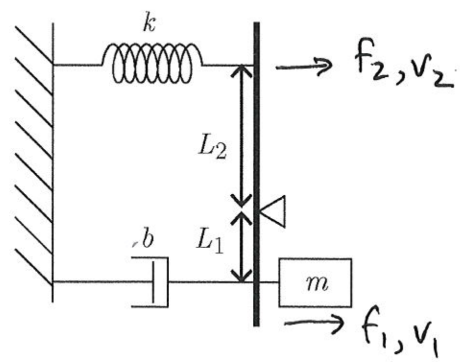


$$x(t) = e^{-3t}(A + Bt) \quad \text{or} \quad e^{-t/\tau}(A + Bt) \longrightarrow \tau = \frac{1}{3}$$

6. (6 pts) Let x be the extension of the spring in the lever-mass-spring-damper system shown below. The second-order differential equation for the system can be written $\ddot{x} + p\dot{x} + qx = 0$. Find p and q in terms of the constants in the figure.

FB: $f_2 = -f_s, \; f_1 = -f_d - m\dot{v}_m, \; f_1 L_1 = f_2 L_2$

GC: $v_d = v_1, \; v_m = v_d, \; v_s = v_2, \; \dfrac{v_1}{L_1} = -\dfrac{v_2}{L_2}$



CL: $f_s = kx, \; f_d = bv_d$

State vars: x, v_m

$$\dot{x} = v_s = v_2 = -v_1 \frac{L_2}{L_1} = -v_m \frac{L_2}{L_1}$$

$$\dot{v}_m = \frac{-f_1 - f_d}{m} \quad \begin{cases} f_1 = f_2 \dfrac{L_2}{L_1} = -f_s \dfrac{L_2}{L_1} = -kx\dfrac{L_2}{L_1} \\ f_d = bv_d = bv_m \end{cases}$$

$$\dot{v}_m = \left(k\frac{L_2}{L_1}x - bv_m\right)\frac{1}{m}$$

$$-\frac{L_1}{L_2}\ddot{x} = \frac{1}{m}\left(k\frac{L_2}{L_1}x + b\frac{L_1}{L_2}\dot{x}\right)$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}\left(\frac{L_2}{L_1}\right)^2 x = 0$$

so

$$p = \frac{b}{m}$$

$$q = \frac{k}{m}\left(\frac{L_2}{L_1}\right)^2$$