ME 449 Notation and Formula Summary Sheet

Chapter 2

- Grübler's formula for the DOF of mechanisms with N links (including ground) and J joints, where joint i has f_i degrees of freedom and $m = 3$ for planar mechanisms or $m = 6$ for spatial mechanisms:

$$\text{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

- Pfaffian velocity constraints take the form $A(\theta)\dot{\theta} = 0$.

Chapter 3

- An element R of $SO(3)$ satisfies $R^T R = I$ and $\det R = 1$, and therefore $R^{-1} = R^T$. Also $R_{ab} = R_{ba}^{-1}$ and $R_{ab}v_b = v_a$, while $R_{ab}v_a = v'_a$, which is the original vector v_a rotated by the rotation that takes $\{a\}$ to $\{b\}$.

- Let R_1 be the orientation achieved when rotating about a fixed axis ω ($\|\omega\| = 1$) a distance θ from an initial orientation $R = I$. Then $R_1 R_a$ is the orientation achieved by rotating $\{a\}$ about ω intrepreted as a space frame angular velocity, while $R_a R_1$ is the orientation achieved by rotating $\{a\}$ about ω interpreted as a body frame angular velocity.

- $\dot{x}(t) = Ax(t)$ has solution $x(t) = e^{At}x_0$. A can be viewed as a constant angular velocity or rigid-body twist (angular and linear velocity), in the body or space frame.

- For $\omega \in \mathbb{R}^3$, we have $\omega \times x = [\omega]x$, where

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

- Rodrigues' formula, integrating a rotation with an angular velocity ω with $\|\omega\| = 1$ for time (or angle) θ: $e^{[\omega]\theta} = I + \sin\theta[\omega] + (1 - \cos\theta)[\omega]^2$. ω and θ together are called the axis-angle representation of an orientation of an element of $SO(3)$, and $\omega\theta \in \mathbb{R}^3$ is the exponential coordinate representation of an an element of $SO(3)$.

- The matrix log of R, in the general case, is given by: $\theta = \cos^{-1}((\text{trace}(R) - 1)/2) \in [0, \pi)$ and $[\omega] = (R - R^T)/(2\sin\theta)$. If $R = I$, then $\theta = 0$. If $\text{trace}(R) = -1$, then $\theta = \pi$. We write $\log(R) = [\omega]\theta$.

- A rigid-body configuration is written $T \in SE(3)$ with the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

where $R \in SO(3)$ and $p \in \mathbb{R}^3$. Also,

$$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix},$$

$T_{ab}T_{bc} = T_{ac}$, $T_{ab}^{-1} = T_{ba}$, and $x_a = T_{ab}x_b$.

- A spatial velocity, or twist, is written $\mathcal{V} = (\omega, v) \in \mathbb{R}^6$, which we can also write in the matrix form

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$$

- Consider a screw motion following the twist $\mathcal{S}' = (\omega', v')$ for duration 1. We can write this as $\mathcal{S}' = \mathcal{S}\theta$, where $\mathcal{S} = (\omega, v)$ and θ is the "distance" of motion along the screw axis $\mathcal{S}$. If $\omega' \neq 0$, then $\mathcal{S} = \mathcal{S}'/\|\omega\|$ and θ is the net rotation about the screw axis. If $\omega' = 0$, then $\mathcal{S} = \mathcal{S}'/\|v'\|$ and θ is the translation along the axis.

  The net displacement obtained by motion along the screw axis $[\mathcal{S}]$ by θ from the identity element of $SE(3)$, in either the body or space frame (since they are initially aligned with each other), is

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2 v \\ 0 & 1 \end{bmatrix}.$$

  For $\omega = 0$, i.e., $\mathcal{S} = (0, v)$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

  For $T = e^{[\mathcal{S}]\theta}$, $\mathcal{S}\theta \in \mathbb{R}^6$ are the exponential coordinates of T.

- The matrix log of $T = (R, p)$, for the general case, is given by

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\operatorname{trace}(R) - 1}{2}\right) \in [0, \pi) \\ [\omega] &= \frac{1}{2\sin\theta}(R - R^T) \\ v &= \left(\frac{1}{\theta}I + \frac{1}{2}[\omega] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})[\omega]^2\right)p. \end{aligned} \tag{0.1}$$

  If $R = I$, then $\omega = 0$, $v = p/\|p\|$,and $\theta = \|p\|$. If $\operatorname{trace}(R) = -1$, then $\theta = \pi$. We write $\log(T) = [\mathcal{S}]\theta$.

- The quantity $T' = e^{[\mathcal{S}]\theta}T$ is the new configuration after T undergoes a screw motion $\mathcal{S}\theta$ in the space frame. The quantity $T' = Te^{[\mathcal{S}]\theta}$ is the new configuration after T undergoes a screw motion $\mathcal{S}\theta$ in the body frame.

- Given frames $\{s\}$ and $\{b\}$, a particular spatial velocity can be represented in these frames as $\mathcal{V}_s$ or $\mathcal{V}_b$, and these are related by the Adjoint transformation
$$\mathcal{V}_s = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b),$$
where $\mathrm{Ad}_{T_{sb}}(\mathcal{V}_b) = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$ and
$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$
The expression $\mathcal{V}_s = \mathrm{Ad}_{T_{sb}}(\mathcal{V}_b)$ is equivalent to $[\mathcal{V}_s] = T_{sb}[\mathcal{V}_b]T_{sb}^{-1}$.

- $\mathrm{Ad}_T^{-1} = \mathrm{Ad}_{T^{-1}}$ and $\mathrm{Ad}_{T_1}(\mathrm{Ad}_{\mathrm{T}_2}(\mathcal{V})) = \mathrm{Ad}_{T_1 T_2}(\mathcal{V})$.

- $\dot{T}T^{-1} = [\mathcal{V}_s]$, the spatial velocity (twist) in space coordinates, and $T^{-1}\dot{T} = [\mathcal{V}_b]$, the spatial velocity (twist) in body coordinates.

- A wrench in space coordinates is written $\mathcal{F}_s = (m_s, f_s) \in \mathbb{R}^6$ and a wrench in body coordinates is written $\mathcal{F}_b = (m_b, f_b)$. $\mathcal{F}_b$ and $\mathcal{F}_s$ are related by
$$\begin{aligned} \mathcal{F}_b &= \mathrm{Ad}^T_{T_{sb}}(\mathcal{F}_s) = [\mathrm{Ad}_{T_{sb}}]^T\mathcal{F}_s \\ \mathcal{F}_s &= \mathrm{Ad}^T_{T_{bs}}(\mathcal{F}_b) = [\mathrm{Ad}_{T_{bs}}]^T\mathcal{F}_b, \end{aligned}$$
derived from the relationship between space and body velocities and the fact that power, $\mathcal{F}_s^T\mathcal{V}_s$ and $\mathcal{F}_b^T\mathcal{V}_b$, must be the same in both frames.

Chapter 4

- The product of exponentials formula for a serial chain manipulator is
$$\begin{aligned} \text{space frame:} \quad & T = e^{[\mathcal{S}_1]\theta_1}\ldots e^{[\mathcal{S}_n]\theta_n}M \\ \text{body frame:} \quad & T = Me^{[\mathcal{B}_1]\theta_1}\ldots e^{[\mathcal{B}_n]\theta_n} \end{aligned}$$
where M is the frame of the end-effector in the space frame when the manipulator is at its home position, $[\mathcal{S}_i]$ is the velocity of the space frame in space coordinates when joint i rotates (or translates) at unit speed while all other joints are fixed, and $[\mathcal{B}_i]$ is the velocity of the body frame in body coordinates when all other joints are fixed.

Chapter 5

- For a manipulator end-effector configuration written in coordinates x, the forward kinematics is $x = f(\theta)$, and the differential kinematics is given by $\dot{x} = \frac{\partial f}{\partial \theta}\dot{\theta} = J(\theta)\dot{\theta}$, where $J(\theta)$ is the manipulator Jacobian.

- In spatial velocities, the relation is $\mathcal{V}_* = J_*(\theta)\dot{\theta}$, where $*$ is either s (space Jacobian) or b (body Jacobian). The columns J_{si} of the space Jacobian are

$$J_{si}(\theta) = \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}}(\mathcal{S}_i)$$

and the columns J_{bi} of the body Jacobian are

$$J_{bi}(\theta) = \mathrm{Ad}_{e^{-[\mathcal{B}_n]\theta_n}\cdots e^{-[\mathcal{B}_{i+1}]\theta_{i+1}}}(\mathcal{B}_i).$$

As expected, the space motion caused by $\mathcal{S}_i$ is only altered by the configurations of joints inboard from joint i (between the joint and the space frame), while the body motion caused by $\mathcal{B}_i$ is only altered by the configurations of joints outboard from joint i (between the joint and the body frame).

The two Jacobians are related by

$$J_b(\theta) = \mathrm{Ad}_{T_{bs}}(\theta)(J_s(\theta)) \quad , \quad J_s(\theta) = \mathrm{Ad}_{T_{sb}}(\theta)(J_b(\theta)).$$

- Generalized forces at the joints τ are related to wrenches expressed in the space or end-effector body frame by

$$\tau = J_*^T(\theta)\mathcal{F}_*,$$

where $*$ is s (space frame) or b (body frame).

- Singularities occur at manipulator configurations where the rank of the Jacobian drops below its maximum value. Often we only care about end-effector motions in a particular subspace, and a singularity is defined when the set of feasible motions in that subspace loses rank.

Chapter 6

- The law of cosines states that $c^2 = a^2 + b^2 - 2ab\cos\gamma$, where a, b, and c are the lengths of the sides of a triangle and γ is the interior angle opposite side c. This formula is often useful to solve inverse kinematics problems.

- Many inverse problems can be stated as finding θ such that $x = f(\theta)$, where x and θ are vectors. Such problems can have many or no solutions, and often admit no closed-form solution. Newton-Raphson iterative numerical root-finding attempts to find a "close by" solution to an initial guess. Starting with an initial guess $\theta(0)$, the iteration is defined by

$$\theta(i+1) = \theta(i) + \left(\frac{\partial f}{\partial \theta}|_{\theta(i)}\right)^{-1} (x - f(\theta(i))),$$

where the expression $x - f(\theta(i))$ is the vector from the current guess to the desired value.

- For inverse kinematics with a desired end-effector configuration $X \in SE(3)$, the direction from the current configuration $T(\theta(i))$ to X, expressed in the end-effector body frame, is given by $[\mathcal{S}] = \log T^{-1}X$. The Newton-Raphson iteration beomes

$$\theta(i+1) = \theta(i) + \underbrace{(J_b(\theta(i))^{-1}\mathcal{S}}_{\Delta\theta_i}.$$

- If the Jacobian is not square (i.e., the number of joints n differs from the degrees of freedom of the end-effector m), then $J_b^{-1}(\theta)$ does not exist. The right generalized inverse $J_b^{-\text{right}} = J_b^T(J_bJ_b^T)^{-1}$ can be used for $n > m$ and the left generalized inverse $J_b^{-\text{left}} = (J_b^TJ_b)^{-1}J_b^T$ can be used for $n < m$.

Chapter 8

- The Lagrangian is the kinetic minus the potential energy, $\mathcal{L}(q,\dot{q}) = K(q,\dot{q}) - U(q)$.

- The Euler-Lagrange equations are

$$\tau = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}.$$

- The kinetic energy of a mechanical system is $K(q,\dot{q}) = \frac{1}{2}\dot{q}^TM(q)\dot{q}$, where M is the mass or inertia matrix.

- The equations of motion of a manipulator can be written

$$\begin{aligned} \tau &= M(\theta)\ddot{\theta} + c(\theta,\dot{\theta}) + \frac{\partial U}{\partial \theta} && (0.2)\\ &= M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + g(\theta) && (0.3)\\ &= M(\theta)\ddot{\theta} + \dot{\theta}^T\Gamma(\theta)\dot{\theta} + g(\theta) && (0.4) \end{aligned}$$

  where $g(\theta)$ are the potential terms (typically due to gravity) and $c(\theta,\dot{\theta})$ is the vector of quadratic velocity terms (Coriolis and centrifugal terms). These quadratic terms are sometimes written as a Coriolis matrix $C(\theta,\dot{\theta})$ multiplied by the linear velocity $\dot{\theta}$, or more insightfully as a quadratic form in terms of the three-dimensional matrix of Christoffel symbols of the mass matrix.

- The Lie bracket of twists $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., the derivative of $\mathcal{V}_2$ in the direction of $\mathcal{V}_1$, is written

$$[\mathcal{V}_1,\mathcal{V}_2] = \mathrm{ad}_{\mathcal{V}_1}(\mathcal{V}_2) = [\mathrm{ad}_{\mathcal{V}_1}]\mathcal{V}_2,$$

  where

$$[\mathrm{ad}_{\mathcal{V}}] = \begin{bmatrix} [\omega] & 0 \\ [v] & [\omega] \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

- The body-frame 6×6 mass matrix of a rigid-body is

$$\mathcal{G}_b = \begin{bmatrix} \mathcal{I}_b & 0 \\ 0 & \mathfrak{m} I \end{bmatrix},$$

  where $\mathcal{I}_b$ is the inertia matrix in the body frame and $\mathfrak{m}$ is the mass.

- The equations of motion of a rigid body, expressed in the body frame, are

$$\mathcal{F}_b = \mathcal{G}_b \dot{\mathcal{V}}_b - [\mathrm{ad}_{\mathcal{V}_b}]^T \mathcal{G}_b \mathcal{V}_b.$$