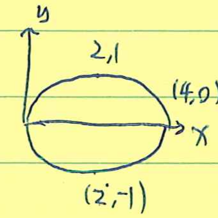
$\frac{55}{56}$ HW4. Chi Zhang.



1. The center of the ellipse is (2,0).

$\therefore$ The function of the path can be written as a function of $\theta \in [0, 2\pi]$: $\begin{cases} x = 2 + 2\cos\theta \\ y = \sin\theta \end{cases}$

As the path goes clockwise from (0,0):

when $s=0$, $\theta = \pi$ when $s = 0.5$, $\theta = 0$; when $s=1$, $\theta = -\pi$.

$\therefore \theta = \pi - 2\pi s$

$$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 + 2\cos(\pi - 2\pi s) \\ \sin(\pi - 2\pi s) \end{bmatrix} = \begin{bmatrix} 2 - 2\cos 2\pi s \\ \sin 2\pi s \end{bmatrix}.$$

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2. $X = \begin{bmatrix} \cos 2\pi s \\ \sin 2\pi s \\ 2s \end{bmatrix}$. $s = 3t^3 + t^2 - 2t + 6$ $\dot{X} = \frac{dX}{ds}\dot{s}$

$\ddot{X} = \frac{dX}{ds}\ddot{s} + \frac{d^2X}{ds^2}\cdot\dot{s}^2$

$$\therefore \frac{dX}{ds} = \begin{bmatrix} -2\pi\sin 2\pi s \\ 2\pi\cos 2\pi s \\ 2 \end{bmatrix} \qquad \frac{d^2X}{ds^2} = \begin{bmatrix} -4\pi^2\cos 2\pi s \\ -4\pi^2 \sin 2\pi s \\ 0 \end{bmatrix}$$

$\dot{s} = 9t^2 + 2t - 2$, $\ddot{s} = 18t + 2$.

$$\therefore \dot{X} = (9t^2+2t-2)\cdot\begin{bmatrix} -2\pi\sin 2\pi(3t^3+t^2-2t+6) \\ 2\pi\cos 2\pi(3t^3+t^2-2t+6) \\ 2 \end{bmatrix}.$$

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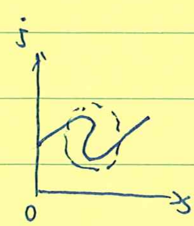
* note that all "+6" term of s in $\dot{X}$ and $\ddot{X}$ can be cancelled directly because $\sin(2\pi\cdot 6 + x) = \sin(12\pi + x) = \sin x$, same as cos.

$$\ddot{X} = (18t+2)\begin{bmatrix} -2\pi\sin 2\pi(3t^3+t^2-2t+6) \\ 2\pi\cos 2\pi(3t^3+t^2-2t+6) \\ 2 \end{bmatrix} - 4\pi^2(9t^2+2t-2)^2\begin{bmatrix} \cos 2\pi(3t^3+t^2-2t+6) \\ \sin 2\pi(3t^3+t^2-2t+6) \\ 0 \end{bmatrix}.$$

21B. Curve A: Impossible. In the circled part, the curve goes back along s axis, which indicates the robot moves back with only forward speed. so it's impossible.



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Curve B: Impossible. It starts from $\dot{s}=0$, but it's not vertical near the initial point. Based on the meaning of this graph, the gradient is $\frac{d\dot{s}}{ds}$. At the ~~initial~~ starting point where $\dot{s}=0$, ds could be nothing but 0, no matter what $d\dot{s}$ is. Thus At this point there are only 2 choices: straight up ~~and~~ or straight down.

Curve C: Possible. The curve is neither going back nor starting with a non-vertical angle. This curve indicates the robot firstly do large accelerating, then reduces the acceleration for a while, and increase acceleration again.

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Point a: Possible, as stated in Curve B. Going straight up means accelerating from the start point a (s is about 0.5, middle of the trajectory), and go forward to the end point; while going straight down in the ~~curve~~ plane means negative accelerating, i.e. start to go back-wards along the trajectory.

Point b and c: Impossible, the reason is stated in Curve B, the direction could be either straight up or down, and won't include the angle between the two direction.

22. (a). Starting from $(s_{lim}, \dot{s}_{test})$, the curve goes along the ~~max~~ min possible acceleration curve, so it is always decelerating till the speed $\dot{s}=0$, then the curve hits s axis, if the curve is not blocked by the velocity limit curve.

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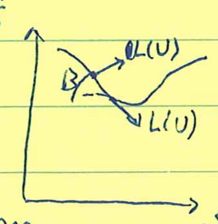
Also, Assume this deceleration curve hits curve F, it means that at the intersection point, the robot switches from max deceleration to max deceleration again, which actually does nothing. In this case, the two curves would be the same, Therefore, they would either never meet each other or coinside.

as the robot is always acting with min acceleration.

(b). As shown on the ~~left~~ right, when the curve touches the velocity limit, the acceleration $\ddot{s}$ can only be L (the same as U). If the curve touches the limit tangentially, the direction $\ddot{s}$ is along the curve, which is the same as the ~~limit~~ L=U direction, so the curve can keep going ~~along~~ forward



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i.e. the direction of the point on the velocity limit boundary.

without ~~touching~~ going into the limit region ; ~~while~~ if the curve hits point B, which is not tangentially hit as shown in the graph, the curve can only go into that region. but it's not admissible, so the robot would go out of the trajectory.

~~In conclusion, only at certain points, the direction of the of $\ddot{s}$ (=L=U) is~~

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(c). At the touching point, the acceleration is limited by L=U, which indicates that minimum and maximum acceleration is actually the same value. Also, after this point, we know that if we keep going along the deceleration curve, we are not maximizing the velocity $\dot{s}$, as we can do max acceleration, considering our goal is maximizing the velocity for time-optimal time-scaling.

This direction points out of the velocity limit region, so if we go along the max acceleration curve, we are going along the same direction, which is the only direction we can go. Also, going along this direction will not bring us back to the speed limit region.

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23. The only modification should be made is start the curve from $(s, \dot{s}) = (0, v_s)$, and ends at $(s, \dot{s}) = (1, v_e)$, where v_s is the starting velocity and v_e is the ending velocity. The process is the same, but the gradient of starting and ending point will not be $+\infty$ or $-\infty$ any longer, but along U or L.

NEED TO CONSIDER FEASIBILITY

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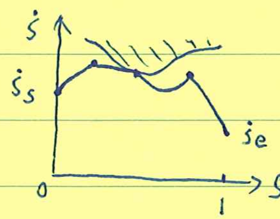
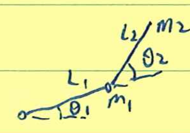


25. First solve the torques τ_1, τ_2 from Lagrangians. Borrowing the result in Ch. 8.1.1:



$$\tau_1 = \begin{bmatrix} (m_1+m_2)L_1^2 + m_2(2L_1L_2\cos\theta_2 + L_2^2) & m_2(L_1L_2\cos\theta_1 + L_2^2) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+\dot{\theta}^T \begin{bmatrix} 0 & -m_2L_1L_2\sin\theta_2 \\ -m_2L_1L_2\sin\theta_2 & -m_2L_1L_2\sin\theta_2 \end{bmatrix} \dot{\theta}$$

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$$\tau_2 = \begin{bmatrix} m_2(L_1L_2\cos\theta_2 + L_2^2) & m_2L_2^2 \end{bmatrix} \ddot{\theta} + \dot{\theta}^T \begin{bmatrix} 0 & -\frac{1}{2}m_2L_1L_2\sin\theta_2 \\ -\frac{1}{2}m_2L_1L_2\sin\theta_2 & 0 \end{bmatrix} \dot{\theta}$$

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$$\theta = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}\frac{\pi}{2}\\ \frac{\pi}{2}\end{bmatrix} s \qquad \dot{\theta} = \frac{d\theta}{ds}\cdot\dot{s} = \begin{bmatrix}\frac{\pi}{2}\\ \frac{\pi}{2}\end{bmatrix}\dot{s} \qquad \ddot{\theta} = \frac{d\theta}{ds}\cdot\ddot{s} + \frac{d^2\theta}{ds^2}\dot{s}^2 = \begin{bmatrix}\frac{\pi}{2}\\ \frac{\pi}{2}\end{bmatrix}\ddot{s}$$

$$\therefore\ \tau_1 = \frac{\pi}{2}\left[(m_1+m_2)L_1^2 + 3m_2L_1L_2\cos(\tfrac{\pi}{2}s) + 2m_2L_2^2\right]\ddot{s} - \frac{\pi^2}{4}\cdot 3m_2L_1L_2\sin(\tfrac{\pi}{2}s)\cdot\dot{s}^2$$

$$\tau_2 = \frac{\pi}{2}m_2\left(L_1L_2\cos(\tfrac{\pi}{2}s) + 2L_2^2\right)\ddot{s} - \frac{\pi^2}{4}m_2L_1L_2\sin(\tfrac{\pi}{2}s)\dot{s}^2 \qquad 4$$

Substitute in $L_1 = 1m$, $L_2 = 0.5m$, $m_1 = 1kg$, $m_2 = 0.5kg$:

$$\tau_1 = \frac{\pi}{2}(1.75 + 0.75\cos(\tfrac{\pi}{2}s))\ddot{s} - \pi^2\cdot 0.1875\sin(\tfrac{\pi}{2}s)\cdot\dot{s}^2$$

$$\tau_2 = \frac{\pi}{4}(0.5\cos(\tfrac{\pi}{2}s) + 0.5)\ddot{s} - \pi^2\cdot 0.0625\sin(\tfrac{\pi}{2}s)\cdot\dot{s}^2$$

$-2 \le \tau_1 \le 2$, $-1 \le \tau_2 \le 1$. Then solve the problem with matlab.

```
clear;
clc;

for s=0:0.2:1
for ds=0:0.2:1
    %Calculate the max and min acceleration of joint1
    U1 = (2 + pi^2*0.1875*sin(pi/2*s)*ds^2) / (pi/2 *
(1.75+0.75*cos(pi/2*s)));
    L1 = (-2 + pi^2*0.1875*sin(pi/2*s)*ds^2) / (pi/2 *
(1.75+0.75*cos(pi/2*s)));

    %Calculate the max and min acceleration of joint2
    U2 = (1 + pi^2*0.0625*sin(pi/2*s)*ds^2) / (pi/2 *
(0.5+0.5*cos(pi/2*s)));
    L2 = (-1 + pi^2*0.0625*sin(pi/2*s)*ds^2) / (pi/2 *
(0.5+0.5*cos(pi/2*s)));

    %Combine 2 joint limits for the total limit of the cone
    U = min(U1,U2);
    L = max(L1,L2);

    %Calculate the length of the vector, uses the scaling of 0.1
    scalingu = 1/sqrt(1^2+(U/ds)^2)*0.1;
    scalingl = 1/sqrt(1^2+(L/ds)^2)*0.1;

    if (ds==0)  %if ds=0, calculate and draw the max and min
acceleration vector
        Ux = [s, s];
        Uy = [0, 0.1];

        Lx = [s, s];
        Ly = [0, -0.1];
        plot(Ux,Uy,Lx,Ly,'linewidth',2);
        hold on;
    else  %else, calculate the max and min acceleration vector
        if (U>=L) % If the cone exists, calculate the upper and lower
bound and draw the cone
            Ux = [s, s+1*scalingu]; %scale the length of the vector to
0.1
            Uy = [ds, ds+U/ds*scalingu];

            Lx = [s, s+1*scalingl];
            Ly = [ds, ds+L/ds*scalingl];

            %Set pamaters for the cone
            ts = atan2(L/ds,1);  %Start of the parameter t
            te = atan2(U/ds,1);  %End of the parameter t
            t = ts:0.01:te;
            xt = s + 0.03*cos(t);  %Draw the cone as an arc with r=0.03
            yt = ds + 0.03*sin(t);
            plot(xt,yt,'k');
            hold on;
            plot(Ux,Uy,Lx,Ly,'linewidth',2);
            hold on;
        end
```

```
    end

end
end

axis equal;
xlabel('S');
ylabel('dS');
title('Cone of S-dS Phase Plane');
legend('Upper bound','Lower bound',4);
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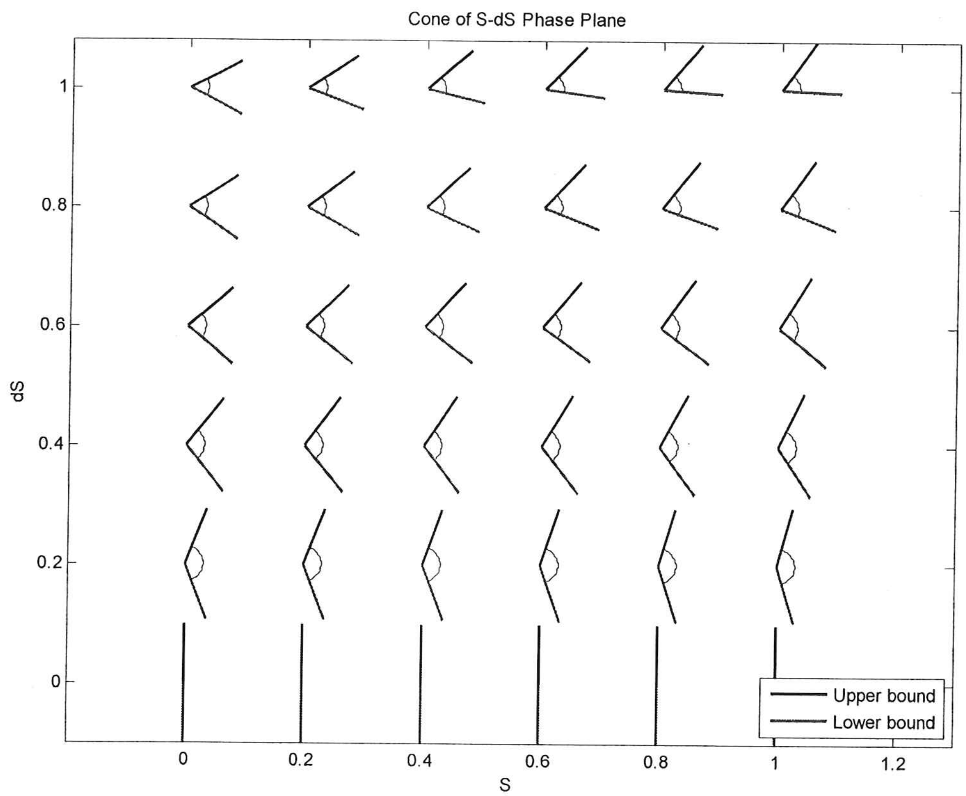

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To solve this problem, we should first derive the dynamics of the system so that we can obtain the equations of torques on both joints. With this specific equation, we can solve the constraints of acceleration based on each joint torque limit when substituting in s for each particular point in the (s, sdot) phase plane. As we only care about the angles of the acceleration cone, I'm scaling each edge's length to 0.1, as shown in the graph.

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