Chapter 2

Configuration Space

A robot is mechanically constructed by connecting a set of bodies, called **links**, to each other using various types of **joints**. **Actuators**, such as electric motors, deliver forces or torques to the joints that cause the robot's links to move. Usually an **end-effector**, such as a gripper or hand for grasping and manipulating objects, is attached to a specific link. All of the robots considered in this book have links that can be modeled as rigid bodies.

Perhaps the most fundamental question one can ask about a robot is, where is it? (In the sense of where the links of the robot are situated.) The answer is the robot's **configuration**: a specification of the positions of all points of the robot. Since the robot's links are rigid and of known shape,[1] only a few numbers are needed to represent the robot's configuration. For example, the configuration of a door can be represented by a single number, the angle θ that the door rotates about its hinge. The configuration of a point lying on a plane can be described by two coordinates, (x, y). The configuration of a coin lying heads up on a flat table can be described by three coordinates: two coordinates (x, y) that specify the location of a particular point on the coin, and one coordinate θ that specifies the coin's orientation. (See Figure 2.1).

The above coordinates all share the common feature of taking values over a continuous range of real numbers. The smallest number of real-valued coordinates needed to represent a robot's configuration is its **degrees of freedom** (**dof**). In the example above, the door (regarded as a robot) has one degree of freedom. The coin lying heads up on a table has three degrees of freedom. Even if the coin could lie either heads up or tails up, its configuration space still would have only three degrees of freedom; introducing a fourth variable to represent which side of the coin is facing up, this variable would then take values in the discrete set $\{\text{heads}, \text{tails}\}$, and not over a continuous range of real values like the other three coordinates (x, y, θ).

Definition 2.1. The **configuration** of a robot is a complete specification of the positions of every point of the robot. The minimum number n of real-

[1] Compare with trying to represent the configuration of a soft object like a pillow.

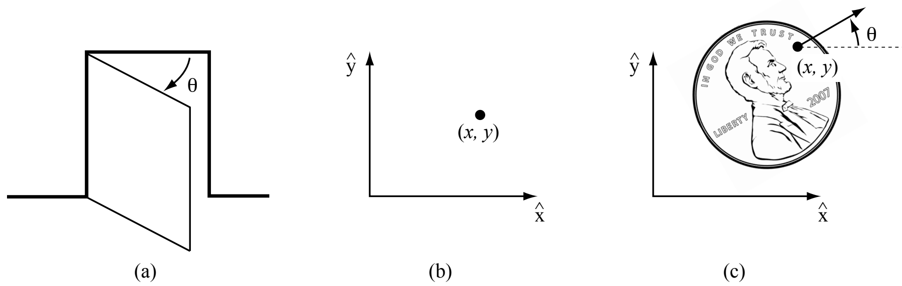


Figure 2.1: (a) The configuration of a door is described by its angle θ. (b) The configuration of a point in a plane is described by coordinates (x, y). (c) The configuration of a coin on a table is described by (x, y, θ).

valued coordinates needed to represent the configuration is the **degrees of freedom** (dof) of the robot. The n-dimensional space containing all possible configurations of the robot is called the **configuration space** (or **C-space**).

In this chapter we study the C-space and degrees of freedom of general robots. Since our robots are constructed of rigid links, we first examine the degrees of freedom of a single rigid body, followed by the degrees of freedom of general multi-link robots. We then study the shape (or topology) and geometry of C-spaces and their mathematical representation. The chapter concludes with a discussion of the C-space of a robot's end-effector, or its **task space**. In the next chapter we study in more detail the various mathematical representations for the C-space of a single rigid body.

2.1 Degrees of Freedom of a Rigid Body

Continuing with the example of the coin lying on the table, choose three points A, B, and C on the coin (Figure 2.2(a)). Once a coordinate frame $\hat{\mathrm{x}}$-$\hat{\mathrm{y}}$ is attached to the plane,[2] the positions of these points in the plane are written (x_A, y_A), (x_B, y_B), and (x_C, y_C). If these points could be placed independently anywhere in the plane, the coin would have six degrees of freedom—two for each of the three points. However, according to the definition of a rigid body, the distance between point A and point B, denoted $d(A, B)$, is always constant regardless of where the coin is. Similarly, the distances $d(B, C)$ and $d(A, C)$ must be constant. The following equality constraints on the coordinates (x_A, y_A), (x_B, y_B), and

[2] The unit axes of coordinate frames are written with a hat, indicating they are unit vectors, in a non-italic font, e.g., $\hat{\mathrm{x}}$, $\hat{\mathrm{y}}$, and $\hat{\mathrm{z}}$.

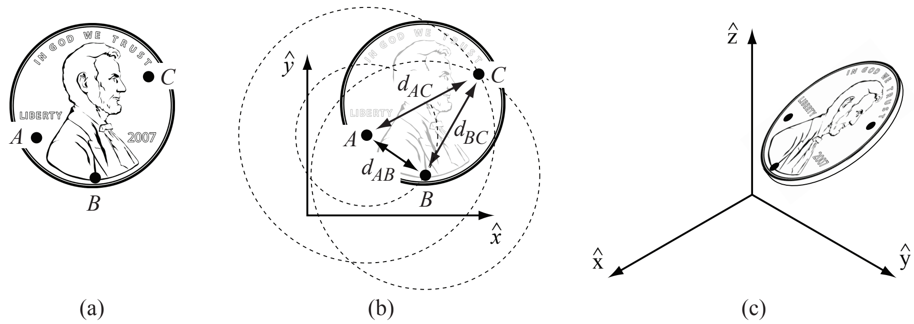


Figure 2.2: (a) Choosing three points fixed to the coin. (b) Once the location of A is chosen, B must lie on a circle of radius d_{AB} centered at A. Once the location of B is chosen, C must lie at the intersection of circles centered at A and B. Only one of these two intersections corresponds to the "heads up" configuration. (c) The configuration of a coin in three-dimensional space is given by the three coordinates of A, two angles to the point B on the sphere of radius d_{AB} centered at A, and one angle to the point C on the circle defined by the intersection of the a sphere centered at A and a sphere centered at B.

(x_C, y_C) must therefore always be satisfied:

$$\begin{aligned} d(A,B) &= \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} = d_{AB} \\ d(B,C) &= \sqrt{(x_B - x_C)^2 + (y_B - y_C)^2} = d_{BC} \\ d(A,C) &= \sqrt{(x_A - x_C)^2 + (y_A - y_C)^2} = d_{AC}. \end{aligned}$$

To determine the number of degrees of freedom of the coin, first choose the position of point A in the plane (Figure 2.2(b)). We may choose it to be anything we want, so we have two degrees of freedom to specify, (x_A, y_A). Once (x_A, y_A) is specified, the constraint $d(A,B) = d_{AB}$ restricts the choice of (x_B, y_B) to those points on the circle of radius d_{AB} centered at A. A point on this circle can be specified by a single parameter, e.g., the angle specifying the location of B on the circle centered at A; let's call this angle ϕ_{AB}, and define it to be the angle that the vector $\overrightarrow{AB}$ makes with the x̂-axis.

Once we have chosen the location of point B, there are only two possible locations for C: at the intersections of the circle of radius d_{AC} centered at A, and the circle of radius d_{BC} centered at B (Figure 2.2(b)). These two solutions correspond to heads or tails. In other words, once we have placed A and B and chosen heads or tails, the two constraints $d(A,C) = d_{AC}$ and $d(B,C) = d_{BC}$ eliminate the two apparent freedoms provided by (x_C, y_C), and the location of C is fixed. The coin has exactly three degrees of freedom in the plane, which can be specified by (x_A, y_A, ϕ_{AB}).

Suppose we were to choose an additional point D on the coin. This new point introduces three additional constraints: $d(A,D) = d_{AD}$, $d(B,D) = d_{BD}$, and $d(C,D) = d_{CD}$. One of these constraints is *redundant*, i.e., it provides no new information; only two of the three constraints are independent. The two freedoms apparently introduced by the coordinates (x_D, y_D) are then immediately eliminated by these two independent constraints. The same would hold for any other newly chosen point on the coin, so that there is no need to consider additional points.

We have been applying the following general rule for determining the number of degrees of freedom of a system:

$$\begin{aligned}\text{Degrees of freedom} = {} & (\text{Sum of freedoms of the points}) - \\ & (\text{Number of independent constraints}).\end{aligned} \tag{2.1}$$

This rule can also be expressed in terms of the number of variables and independent equations that describe the system:

$$\begin{aligned}\text{Degrees of freedom} = {} & (\text{Number of variables}) - \\ & (\text{Number of independent equations}).\end{aligned} \tag{2.2}$$

This general rule can also be used to determine the number of freedoms of a rigid body in three dimensions. For example, assume our coin is no longer confined to the table (Figure 2.2(c)). The coordinates for the three points A, B, and C are now given by (x_A, y_A, z_A), (x_B, y_B, z_B), and (x_C, y_C, z_C), respectively. Point A can be placed freely (three degrees of freedom). The location of point B is subject to the constraint $d(A, B) = d_{AB}$, meaning it must lie on the sphere of radius d_{AB} centered at A. Thus we have $3-1 = 2$ freedoms to specify, which can be expressed as the latitude and longitude for the point on the sphere. Finally, the location of point C must lie at the intersection of spheres centered at A and B of radius d_{AC} and d_{BC}, respectively. In the general case the intersection of two spheres is a circle, and the location of point C can be described by an angle that parametrizes this circle. Point C therefore adds $3 - 2 = 1$ freedom. Once the position of point C is chosen, the coin is fixed in space.

In summary, a rigid body in three-dimensional space has six freedoms, which can be described by the three coordinates parametrizing point A, the two angles parametrizing point B, and one angle parametrizing point C. Other representations for the configuration of a rigid body are discussed in Chapter 3.

We have just established that a rigid body moving in three-dimensional space, which we call a **spatial rigid body**, has six degrees of freedom. Similarly, a rigid body moving in a two-dimensional plane, which we henceforth call a **planar rigid body**, has three degrees of freedom. This latter result can also be obtained by considering the planar rigid body to be a spatial rigid body with six degrees of freedom, but with the three independent constraints $z_A = z_B = z_C = 0$.

Since our robots are constructed of rigid bodies, Equation (2.1) can be expressed as follows:

$$\begin{aligned}\text{Degrees of freedom} = {} & (\text{Sum of freedoms of the bodies}) - \\ & (\text{Number of independent constraints}).\end{aligned} \tag{2.3}$$

Equation (2.3) forms the basis for determining the degrees of freedom of general robots, which is the topic of the next section.

2.2 Degrees of Freedom of a Robot

Consider once again the door example of Figure 2.1(a), consisting of a single rigid body connected to the wall by a hinge joint. From the previous section we know that the door has only one degree of freedom, conveniently represented by the hinge joint angle θ. Without the hinge joint, the door is free to move in three-dimensional space and has six degrees of freedom. By connecting the door to the wall via the hinge joint, five independent constraints are imposed on the motion of the door, leaving only one independent coordinate (θ). Alternatively, the door can be viewed from above and regarded as a planar body, which has three degrees of freedom. The hinge joint then imposes two independent constraints, again leaving only one independent coordinate (θ). Its C-space is represented by some range in the interval $[0, 2\pi)$ over which θ is allowed to vary.

In both cases the joints have the effect of constraining the motion of the rigid body, and thus reducing the overall degrees of freedom. This observation suggests a formula for determining the number of degrees of freedom of a robot, simply by counting the number of rigid bodies and joints. In this section we derive precisely such a formula, called Grübler's formula, for determining the number of degrees of freedom of planar and spatial robots.

2.2.1 Robot Joints

Figure 2.3 illustrates the basic joints found in typical robots. Every joint connects exactly two links; joint that simultaneously connect three or more links are not allowed. The **revolute joint** (R), also called a hinge joint, allows for rotational motion about the joint axis. The **prismatic joint** (P), also called a sliding or linear joint, allows for translational (or rectilinear) motion along the direction of the joint axis. The **screw joint** (H), also called a helical joint, allows simultaneous rotation and translation about a screw axis. Revolute, prismatic, and screw joints all have one degree of freedom.

Joints can also have multiple degrees of freedom. The **cylindrical joint** (C) is a two-dof joint that allows for independent translations and rotations about a single fixed joint axis. The **universal joint** (U) is another two-dof joint that consists of a pair of revolute joints arranged so that their joint axes are orthogonal. The **spherical joint** (S), also called a ball-and-socket joint, has three degrees of freedom and functions much like our shoulder joint.

A joint can be viewed as providing freedoms to allow one rigid body to move relative to another. It can also be viewed as providing constraints on the possible motions of the two rigid bodies it connects. For example, a revolute joint can be viewed as allowing one freedom of motion between two rigid bodies in space, or it can be viewed as providing five constraints on the motion of one rigid body relative to the other. Generalizing, the number of degrees of freedom

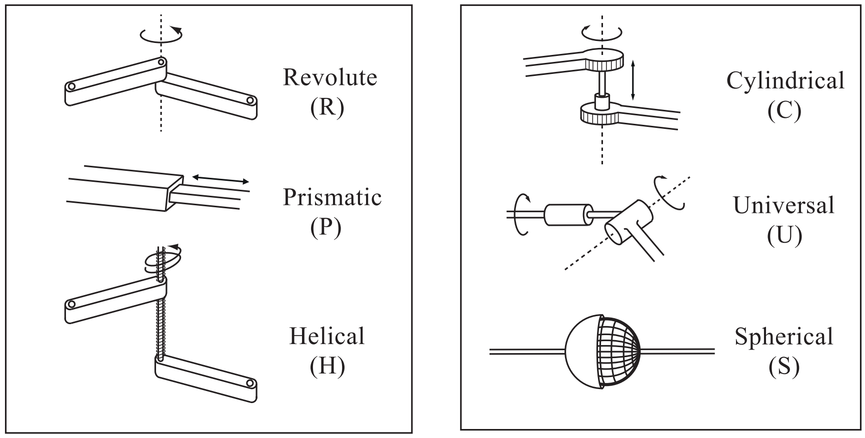


Figure 2.3: Typical robot joints.

Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Screw (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

Table 2.1: The number of degrees of freedom and constraints provided by common joints.

of a rigid body (three for planar bodies and six for spatial bodies) minus the number of constraints provided by a joint must equal the number of freedoms provided by the joint.

The freedoms and constraints provided by the various joint types are summarized in Table 2.1.

2.2.2 Grübler's Formula

The number of degrees of freedom of a mechanism with links and joints can be calculated using **Grübler's formula**, which is an expression of Equation (2.3).

Proposition 2.1. *Consider a mechanism consisting of N links, where ground is also regarded as a link. Let J be the number of joints, m be the number of degrees of freedom of a rigid body ($m = 3$ for planar mechanisms and $m = 6$ for*

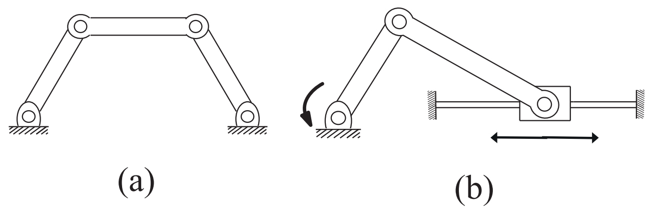


Figure 2.4: (a) Four-bar linkage. (b) Slider-crank mechanism.

spatial mechanisms), f_i be the number of freedoms provided by joint i, and c_i be the number of constraints provided by joint i (it follows that $f_i + c_i = m$ for all i). Then Grübler's formula for the degrees of freedom (dof) of the robot is

$$\begin{aligned}\text{dof} &= \underbrace{m(N-1)}_{\text{rigid body freedoms}} - \underbrace{\sum_{i=1}^{J} c_i}_{\text{joint constraints}} \\ &= m(N-1) - \sum_{i=1}^{J}(m - f_i) \\ &= m(N-1-J) + \sum_{i=1}^{J} f_i. \qquad (2.4)\end{aligned}$$

This formula only holds if all joint constraints are independent. If they are not independent, then the formula provides a lower bound on the number of degrees of freedom.

Below we apply Grübler's formula to several planar and spatial mechanisms. We distinguish between two types of mechanisms: **open-chain mechanisms** (also known as **serial mechanisms**) and **closed-chain mechanisms**. A closed-chain mechanism is any mechanism that has a closed loop. A person standing with both feet on the ground is an example of a closed-chain mechanism, since a closed loop is traced from the ground, through the right leg, through the waist, through the left leg, and back to the ground (recall that ground itself is a link). An open-chain mechanism is any mechanism without a closed loop; an example is your arm when your hand is allowed to move freely in space.

Example 2.1. Four-bar linkage and slider-crank mechanism
The planar four-bar linkage shown in Figure 2.4(a) consists of four links (one of them ground) arranged in a single closed loop and connected by four revolute joints. Since all the links are confined to move in the same plane, $m = 3$. Subsituting $N = 4$, $J = 4$, and $f_i = 1$, $i = 1, \ldots, 4$, into Grübler's formula, we see that the four-bar linkage has one degree of freedom.

The slider-crank closed-chain mechanism of Figure 2.4(b) can be analyzed in two ways: (i) the mechanism consists of three revolute joints and one prismatic

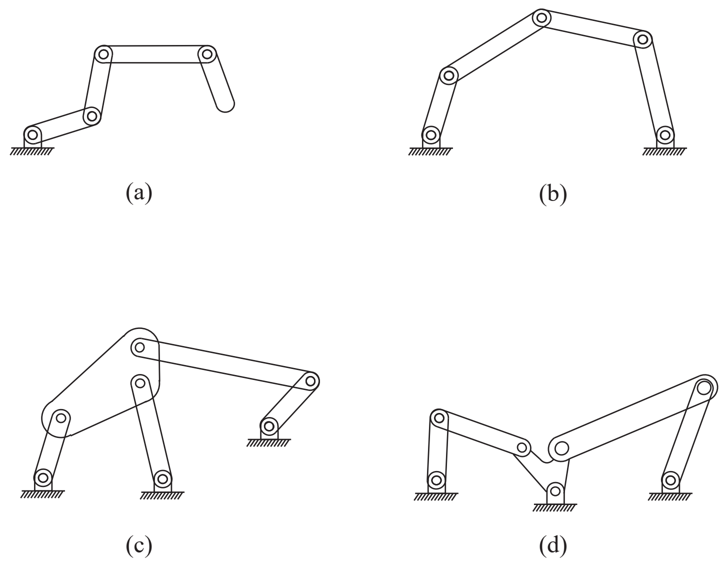


Figure 2.5: (a) k-link planar serial chain. (b) Five-bar planar linkage. (c) Stephenson six-bar linkage. (d) Watt six-bar linkage.

joint ($J = 4$, and each $f_i = 1$) and four links ($N = 4$, including the ground link), or (ii) the mechanism consists of two revolute joints ($f_i = 1$) and one RP joint (the RP joint is a concatenation of a revolute and prismatic joint, so that $f_i = 2$) and three links ($N = 3$; remember that each joint connects precisely two bodies). In both cases the mechanism has one degree of freedom.

Example 2.2. Some classical planar mechanisms
Let us now apply Grübler's formula to several classical planar mechanisms. The k-link planar serial chain of revolute joints in Figure 2.5(a) (called a kR robot for its k revolute joints) has $N = k + 1$ (k links plus ground) and $J = k$, and since all the joints are revolute, each $f_i = 1$. Therefore,

$$\mathrm{dof} = 3((k + 1) - 1 - k) + k = k$$

as expected. For the planar five-bar linkage of Figure 2.5(b), $N = 5$ (four links plus ground), $J = 5$, and since all joints are revolute, each $f_i = 1$. Therefore,

$$\mathrm{dof} = 3(5 - 1 - 5) + 5 = 2.$$

For the Stephenson six-bar linkage of Figure 2.5(c), we have $N = 6$, $J = 7$, and $f_i = 1$ for all i, so that

$$\mathrm{dof} = 3(6 - 1 - 7) + 7 = 1.$$

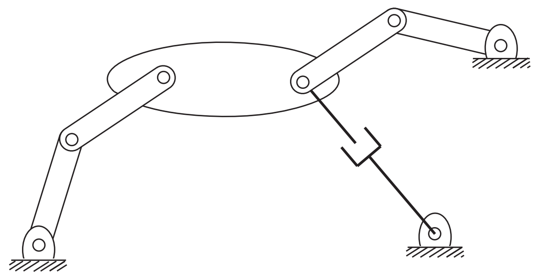


Figure 2.6: A planar mechanism with two overlapping joints.

Finally, for the Watt six-bar linkage of Figure 2.5(d), we have $N = 6$, $J = 7$, and $f_i = 1$ for all i, so that like the Stephenson six-bar linkage,

$$\text{dof} = 3(6 - 1 - 7) + 7 = 1.$$

Example 2.3. A planar mechanism with overlapping joints
The planar mechanism illustrated in Figure 2.6 has three links that meet at a single point on the right of the large link. Recalling that a joint by definition connects exactly two links, the joint at this point of intersection should not be regarded as a single revolute joint. Rather, it is correctly interpreted as two revolute joints overlapping each other. Again, there is more than one way to derive the number of degrees of freedom of this mechanism using Grübler's formula: (i) The mechanism consists of eight links ($N = 8$), eight revolute joints, and one prismatic joint. Substituting into Grübler's formula,

$$\text{dof} = 3(8 - 1 - 9) + 9(1) = 3.$$

(ii) Alternatively, the lower-right revolute-prismatic joint pair can be regarded as a single two-dof joint. In this case the number of links is $N = 7$, with seven revolute joints, and a single two-dof revolute-prismatic pair. Substituting into Grübler's formula,

$$\text{dof} = 3(7 - 1 - 8) + 7(1) + 1(2) = 3.$$

Example 2.4. Grübler's formula and singularities
For the parallelogram linkage of Figure 2.7(a), $N = 5$, $J = 6$, and $f_i = 1$ for each joint. From Grübler's formula, the degrees of freedom is given by $3(5 - 1 - 6) + 6 = 0$. A mechanism with zero degrees of freedom is by definition a rigid structure. However, if the three parallel links are of the same length, and the two horizontal rows of joints are also collinear as implied by the figure, the mechanism can in fact move with one degree of freedom.

A similar situation occurs for the two-dof planar five-bar linkage of Figure 2.7(b). If the two joints connected to ground are locked at some fixed angle,

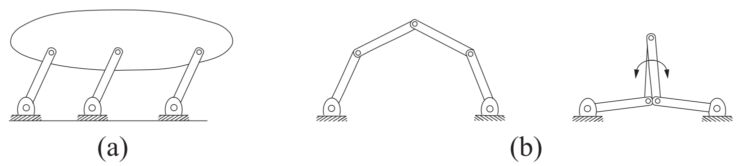


Figure 2.7: (a) A parallelogram linkage; (b) The five-bar linkage in a regular and singular configuration.

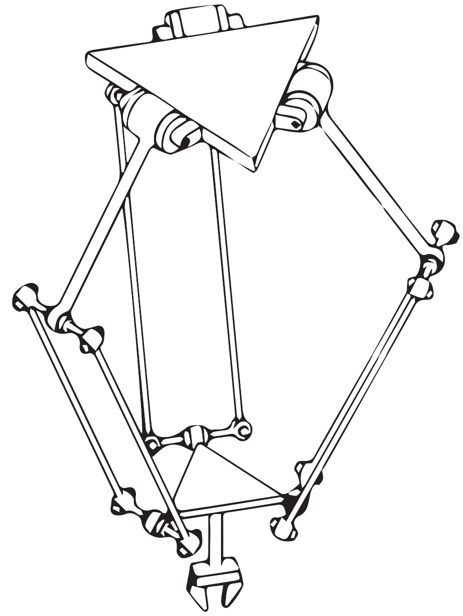


Figure 2.8: The Delta robot.

the five-bar linkage should then become a rigid structure. However, if the two middle links are of equal length and overlap each other as illustrated in the figure, then these overlapping links can rotate freely about the two overlapping joints. Of course, the link lengths of the five-bar linkage must meet certain specifications in order for such a configuration to even be possible. Also note that if a different pair of joints is locked in place, the mechanism does become a rigid structure as expected.

Grübler's formula provides a lower bound on the degrees of freedom for singular cases like those just described. Configuration space singularities arising in closed chains are discussed in Chapter 7.

Example 2.5. Delta robot
The Delta robot of Figure 2.8 consists of two platforms—the lower one mobile,

the upper one stationary—connected by three legs: each leg consists of an RR serial chain connected to a closed-loop parallelogram linkage. Naively applying the spatial version of Grübler's formula yields $N = 17$, $J = 21$ (all revolute joints), and dof $= 17(17 - 1 - 21) + 21 = -9$, which would imply that the mechanism is overconstrained and therefore a rigid structure. However, each parallelogram linkage has one degree of freedom of motion, so that each leg of the Delta robot is kinematically equivalent to an RUU chain. In this case $N = 8$, $J = 9$ (six U joints and three R joints), and

$$\text{dof} = 6(8 - 1 - 9) + 3(1) + 6(2) = 3.$$

The Delta robot has three degrees of freedom. The moving platform in fact always remains parallel to the fixed platform and in the same orientation, so that the Delta robot in effect acts as a Cartesian positioning device.

Example 2.6. Stewart-Gough Platform
The Stewart-Gough platform of Figure 1.1(b) consists of two platforms—the lower one stationary and regarded as ground, the upper one mobile—connected by six universal-prismatic-spherical (UPS) serial chains. The total number of links is fourteen ($N = 14$). There are six universal joints (each with two degrees of freedom, $f_i = 2$), six prismatic joints (each with a single degree of freedom, $f_i = 1$), and six spherical joints (each with three degrees of freedom, $f_i = 3$). The total number of joints is 18. Substituting these values into Grübler's formula with $m = 6$,

$$\text{dof} = 6(14 - 1 - 18) + 6(1) + 6(2) + 6(3) = 6.$$

In some versions of the Stewart-Gough platform the six universal joints are replaced by spherical joints. By Grübler's formula this mechanism would have twelve degrees of freedom; replacing each universal joint by a spherical joint introduces an extra degree of freedom in each leg, allowing torsional rotations about the leg axis. Note however that this torsional rotation has no effect on the motion of the mobile platform.

The Stewart-Gough platform is a popular choice for car and airplane cockpit simulators, as the platform moves with the full six degrees of freedom of motion of a rigid body. The parallel structure means that each leg only needs to support a fraction of the weight of the payload. On the other hand, the parallel design also limits the range of translational and rotational motion of the platform.

2.3 Configuration Space: Topology and Representation

2.3.1 Configuration Space Topology

Until now we have been focusing on one important aspect of a robot's C-space—its dimension, or the number of degrees of freedom. However, the *shape* of the space is also important.

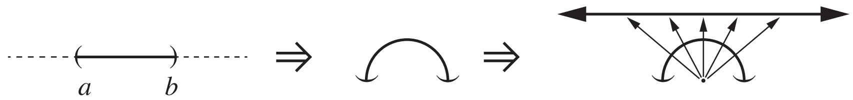


Figure 2.9: An open interval of the real line, denoted (a, b), can be deformed to an open semicircle. This open semicircle can then be deformed to the real line by the mapping illustrated: beginning from a point at the center of the semicircle, draw a ray that intersects the semicircle and then a line above the semicircle. These rays show that every point of the semicircle can be stretched to exactly one point on the line, and vice-versa. Thus an open interval can be continuously deformed to a line, so an open interval and a line are topologically equivalent.

Consider a point moving on the surface of a sphere. The point's C-space is two-dimensional, as the configuration can be described by two coordinates, e.g., latitude and longitude. As another example, a point moving on a plane also has a two-dimensional C-space, with coordinates (x, y). While both a plane and the surface of a sphere are two-dimensional, clearly they do not have the same shape—the plane extends infinitely while the sphere wraps around.

On the other hand, a larger sphere has the same shape as the original sphere, in that it wraps around in the same way. Only its size is different. For that matter, an oval-shaped American football also wraps around similarly to a sphere. The only difference between a football and a sphere is that the football has been stretched in one direction.

The idea that the two-dimensional surfaces of a small sphere, a large sphere, and a football all have the same kind of shape, which is different from the shape of a plane, is expressed by the **topology** of the surfaces. We do not attempt a rigorous treatment in this book[3], but we say that two spaces are **topologically equivalent** if one can be continuously deformed into the other without cutting or gluing. A sphere can be deformed into a football simply by stretching, without cutting or gluing, so those two spaces are topologically equivalent. You cannot turn a sphere into a plane without cutting it, however, so a sphere and a plane are not topologically equivalent.

Topologically distinct one-dimensional spaces include the circle, the line, and a closed interval of the line. The circle is written mathematically as S or S^1, i.e., a one-dimensional "sphere." The line can be written $\mathbb{E}$ or $\mathbb{E}^1$, indicating a one-dimensional Euclidean (or "flat") space. Since a point in $\mathbb{E}^1$ is so commonly represented by a real number (after choosing an origin and a length scale), it is often written $\mathbb{R}$ or $\mathbb{R}^1$ instead. A closed interval of the line, which contains its endpoints, can be written $[a, b] \subset \mathbb{R}^1$. (An open interval (a, b) does not include the endpoints a and b and is topologically equivalent to a line, since the open interval can be stretched to a line, as shown in Figure 2.9. A closed interval is not topologically equivalent to a line, since a line does not contain endpoints.)

[3]For those familiar with concepts in topology, all spaces we consider can be viewed as embedded in a higher-dimensional Euclidean space, inheriting the Euclidean topology of that

In higher dimensions, $\mathbb{R}^n$ is the n-dimensional Euclidean space and S^n is the n-dimensional surface of a sphere in $(n+1)$-dimensional space. For example, S^2 is the two-dimensional surface of a sphere in three-dimensional space.

Note that the topology of a space is a fundamental property of the space itself, and *it is independent of how we choose to use coordinates to represent points in the space*. For example, to represent a point on a circle, we could refer to the point by the angle θ from the center of the circle to the point, relative to a chosen zero angle. Or we could choose a reference frame with its origin at the center of the circle and represent the point by the two coordinates (x, y), subject to the constraint $x^2 + y^2 = 1$. No matter our choice of coordinates, the space itself does not change.

Some C-spaces can be expressed as the **Cartesian product** of two or more spaces of lower dimension; that is, points in such a C-space can be represented as the union of the representation of points in the lower-dimensional spaces. For example:

- The C-space of a rigid body in the plane can be written as $\mathbb{R}^2 \times S^1$, since the configuration can be represented as the concatenation of the coordinates (x, y) representing $\mathbb{R}^2$ and an angle θ representing S^1.
- The C-space of a PR robot arm can be written $\mathbb{R}^1 \times S^1$. (Typically we will not worry about joint limits when expressing the topology of the C-space.)
- The C-space of a 2R robot arm can be written as $S^1 \times S^1 = T^2$, where T^n is the n-dimensional surface of a torus in an $(n+1)$-dimensional space. (See Table 2.2.) Note that $S^1 \times S^1 \times \ldots \times S^1$ (n copies of S^1) is equal to T^n, not S^n; for example, a sphere S^2 is not topologically equivalent to a torus T^2.
- The C-space of a planar rigid body (e.g., the chassis of a mobile robot) with a 2R robot arm can be written as $\mathbb{R}^2 \times S^1 \times T^2 = \mathbb{R}^2 \times T^3$.
- As we saw in Section 2.1 when we counted the degrees of freedom of a rigid body in three dimensions, the configuration of a rigid body can be described by a point in $\mathbb{R}^3$, plus a point on a two-dimensional sphere S^2, plus a point on a one-dimensional circle S^1, for a total C-space of $\mathbb{R}^3 \times S^2 \times S^1$.

2.3.2 Configuration Space Representation

To perform computations, we must have a numerical *representation* of the space, consisting of a set of real numbers. We are familiar with this idea from linear algebra—a vector is a natural way to represent a point in a Euclidean space. It is important to keep in mind that the representation of a space involves a choice, and therefore it is not as fundamental as the topology of the space itself,

---

space.

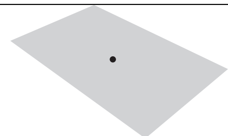
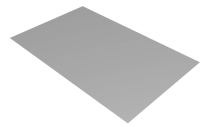
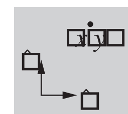
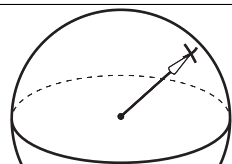
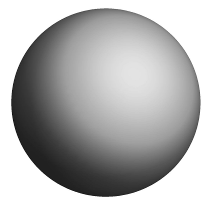
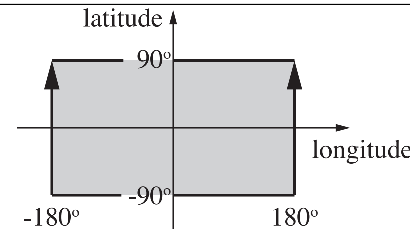
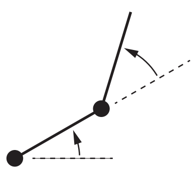
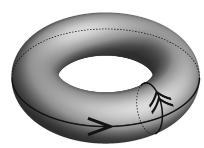
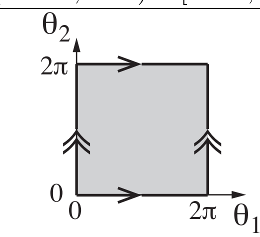
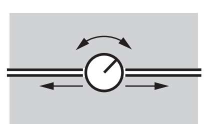
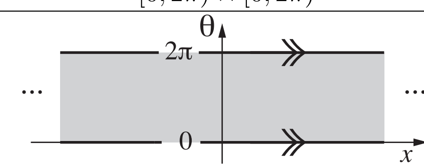
system	C-space	sample representation
 point on a plane	 $\mathbb{E}^2$	 $\mathbb{R}^2$
 tip of spherical pendulum	 2-sphere S^2	 $[-180°, 180°) \times [-90°, 90°]$
 2R robot arm	 2-torus $T^2 = S^1 \times S^1$	 $[0, 2\pi) \times [0, 2\pi)$
 rotating sliding knob	 cylinder $\mathbb{R}^1 \times S^1$	 $\mathbb{R}^1 \times [0, 2\pi)$

Table 2.2: Four topologically different two-dimensional C-spaces and example coordinate representations. In the latitude-longitude representation of the sphere, the latitudes $-90°$ and $90°$ each correspond to a single point (the South Pole and the North Pole, respectively), and the longitude parameter wraps around at $180°$ and $-180°$: the edges with the arrows are glued together. Similarly, the coordinate representations of the torus and cylinder wrap around at the edges marked with identical arrows. To turn the torus into its coordinate representation (a subset of $\mathbb{R}^2$), the torus can be cut along the small circle shown (representing the range of angles θ_2 of the second joint while $\theta_1 = 0$) and straightened out to make a cylinder, then cut along the length of the cylinder (representing the range of angles of the first joint while $\theta_2 = 0$) and flattened.

which is independent of the representation. For example, the same point in a 3D space can have different coordinate representations depending on the choice of the reference frame (the origin and the direction of the coordinate axes) and the choice of length scale, but the topology of the underlying space is the same regardless of our choice.

While it is natural to choose a reference frame and length scale and use a vector to represent points in a Euclidean space, representing a point on a curved space, like a sphere, is less obvious. One solution for a sphere is to use latitude and longitude coordinates. A choice of n coordinates, or parameters, to represent an n-dimensional space is called an **explicit parametrization** of the space. The explicit parametrization is valid for a particular range of the parameters (e.g., $[-90°, 90°]$ for latitude and $[-180°, 180°)$ for longitude for a sphere, where, on Earth, negative values correspond to "South" and "West," respectively).

The latitude-longitude representatation of a sphere is dissatisfying if you are walking near the North Pole (latitude equals 90°) or South Pole (latitude equals −90°), where taking a very small step can result in a large change in the coordinates. The North and South Poles are *singularities* of the representation, and the existence of singularities is a result of the fact that a sphere does not have the same topology as a plane, i.e., the space of the two real numbers that we have chosen to represent the sphere (latitude and longitude). The location of these singularities has nothing to do with the sphere itself, which looks the same everywhere, and everything to do with the chosen representation of it. Singularities of the parametrization are particularly problematic when representing velocities as the time rate of change of coordinates, since these representations may tend to infinity near singularities even if the point on the sphere is moving at a constant speed $\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ (if you had represented the point as (x, y, z) instead).

If you assume that the configuration never approaches a singularity of the representation, you can ignore this issue. If you cannot make this assumption, there are two ways to overcome the problem:

- Define more than one **coordinate chart** on the space, where each coordinate chart is an explicit parametrization covering only a portion of the space. Within each chart, there is no singularity. For example, we could define two coordinate charts on the sphere: the usual latitude $\phi \in [-90°, 90°]$ and longitude $\psi \in [-180°, 180°)$, and alternative coordinates (ϕ', ψ') in a rotated coordinate frame, where the alternative latitude ϕ' is 90° at the "East Pole" and −90° at the "West Pole." Then the first coordinate chart can be used when $-90° + \epsilon < \phi < 90° - \epsilon$, for some small $\epsilon > 0$, and the second coordinate chart can be used when $-90° + \epsilon < \phi' < 90° - \epsilon$.

  If we define a set of singularity-free coordinate charts that overlap each other and cover the entire space, like the two charts above, the charts are said to form an **atlas** of the space, much like an atlas of the Earth consists of several maps that together cover the Earth. An advantage of using an atlas of coordinate charts is that the representation always uses the minimum number of numbers. A disadvantage is the extra bookkeeping required to switch the representation between coordinate charts to avoid singularities. (Note that Euclidean spaces can be covered by a single coordinate chart without singularities.)

- Instead of using an explicit parametrization, use an **implicit representation** of the space. An implicit representation views the n-dimensional space as embedded in a Euclidean space of more than n dimensions, just like a two-dimensional unit sphere can be viewed as a surface embedded in a three-dimensional Euclidean space. An implicit representation uses the coordinates of the higher-dimensional space (e.g., (x, y, z) in the three-dimensional space), but subjects these coordinates to constraints that reduce the number of degrees of freedom (e.g., $x^2 + y^2 + z^2 = 1$ for the unit sphere).

  A disadvantage of this approach is that the representation has more numbers than the number of degrees of freedom. An advantage is that there are no singularities in the representation—a point moving smoothly around the sphere is represented by a smoothly changing (x, y, z), even at the North and South Poles. A single representation is used for the whole sphere; multiple coordinate charts are not needed.

  Another advantage is that while it may be very difficult to construct an explicit parametrization, or atlas, for a closed-chain mechanism, it is easy to find an implicit representation: the set of all joint coordinates subject to the **loop-closure equations** that define the closed loops (Section 2.4).

  We use implicit representations throughout the book, beginning in the next chapter. In particular, we use nine numbers, subject to six constraints, to represent the three orientation freedoms of a rigid body in space. This is called a *rotation matrix*. In addition to being singularity-free (unlike three-parameter representations such as roll-pitch-yaw angles[4]), the rotation matrix representation has the benefit of allowing us to use linear algebra to perform computations such as (1) rotating a rigid body or (2) changing the reference frame in which the orientation of a rigid body is expressed.[5]

In summary, the non-Euclidean shape of many C-spaces motivates the use of implicit representations of C-space throughout this book. We return to this topic in the next chapter.

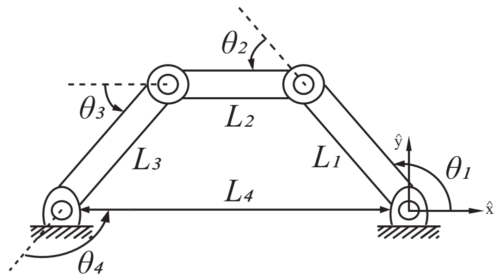


Figure 2.10: The four-bar linkage.

2.4 Configuration and Velocity Constraints

For robots containing one or more closed loops, usually an implicit representation is more easily obtained than an explicit parametrization. For example, consider the planar four-bar linkage of Figure 2.10, which has one degree of freedom. The fact that the four links always form a closed loop can be expressed in the form of the following three equations:

$$\begin{aligned} L_1 \cos\theta_1 + L_2 \cos(\theta_1+\theta_2) + \ldots + L_4 \cos(\theta_1 + \ldots + \theta_4) &= 0 \\ L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) + \ldots + L_4 \sin(\theta_1 + \ldots + \theta_4) &= 0 \\ \theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi &= 0. \end{aligned}$$

These equations are obtained by viewing the four-bar linkage as a serial chain with four revolute joints, in which (i) the tip of link L_4 always coincides with the origin and (ii) the orientation of link L_4 is always horizontal.

These equations are sometimes referred to as **loop-closure equations**. For the four-bar linkage they are given by a set of three equations in four unknowns. The set of all solutions forms a one-dimensional curve in the four-dimensional joint space and constitutes the C-space.

For general robots containing one or more closed loops, the configuration space can be implicitly represented by $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ and loop-closure equations of the form

$$g(\theta) = \begin{bmatrix} g_1(\theta_1, \ldots, \theta_n) \\ \vdots \\ g_k(\theta_1, \ldots, \theta_n) \end{bmatrix} = 0, \qquad (2.5)$$

[4] Roll-pitch-yaw angles and *Euler* angles use three parameters for the space of rotations $S^2 \times S^1$ (two for S^2 and one for S^1), and therefore are subject to singularities as discussed above.

[5] Another singularity-free implicit representation of orientations, the unit quaternion, uses only four numbers subject to the constraint that the four-vector be unit length. In fact, this representation is a double covering of the set of orientations: for every orientation, there are two unit quaternions.

where $g : \mathbb{R}^n \to \mathbb{R}^k$ is a set of k independent equations, with $k \leq n$. Such constraints are known as **holonomic constraints**, constraints that reduce the dimension of the C-space.[6] The C-space can be viewed as a surface of dimension $n - k$ (assuming all constraints are independent) embedded in $\mathbb{R}^n$.

Suppose a closed-chain robot with loop-closure equations $g(\theta) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^k$, is in motion, following the time trajectory $\theta(t)$. Differentiating both sides of $g(\theta(t)) = 0$ with respect to t, we obtain

$$\begin{aligned}
\frac{d}{dt} g(\theta(t)) &= 0 \\
\begin{bmatrix} \frac{\partial g_1}{\partial \theta_1}(\theta)\dot{\theta}_1 + \ldots + \frac{\partial g_1}{\partial \theta_n}(\theta)\dot{\theta}_n \\ \vdots \\ \frac{\partial g_k}{\partial \theta_1}(\theta)\dot{\theta}_1 + \ldots + \frac{\partial g_k}{\partial \theta_n}(\theta)\dot{\theta}_n \end{bmatrix} &= 0 \\
\begin{bmatrix} \frac{\partial g_1}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_1}{\partial \theta_n}(\theta) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_k}{\partial \theta_n}(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} &= 0 \\
\frac{\partial g}{\partial \theta}(\theta)\dot{\theta} &= 0. \qquad (2.6)
\end{aligned}$$

Here $\dot{\theta}_i$ denotes the time derivative of θ_i with respect to time t, $\frac{\partial g}{\partial \theta}(\theta) \in \mathbb{R}^{k \times n}$, and $\theta, \dot{\theta} \in \mathbb{R}^n$. From the above we see that the joint velocity vector $\dot{\theta} \in \mathbb{R}^n$ cannot be arbitrary, but must always satisfy

$$\frac{\partial g}{\partial \theta}(\theta)\dot{\theta} = 0. \qquad (2.7)$$

These constraints can be written in the form

$$A(\theta)\dot{\theta} = 0, \qquad (2.8)$$

where $A(\theta) \in \mathbb{R}^{k \times n}$. Velocity constraints of this form are called **Pfaffian constraints**. For the case of $A(\theta) = \frac{\partial g}{\partial \theta}(\theta)$, one could regard $g(\theta)$ as being the "integral" of $A(\theta)$; for this reason, holonomic constraints of the form $g(\theta) = 0$ are also called **integrable constraints**—the velocity constraints that they imply can be integrated to give equivalent configuration constraints.

We now consider another class of Pfaffian constraints that is fundamentally different from the holonomic type. To illustrate with a concrete example, consider an upright coin of radius r rolling on the plane as shown in Figure 2.11. The configuration of the coin is given by the contact point (x, y) on the plane, the steering angle ϕ, and the angle of rotation (see Figure 2.11). The C-space of the coin is therefore $\mathbb{R}^2 \times T^2$, where T^2 is the two-dimensional torus parametrized by the angles ϕ and θ. This C-space is four-dimensional.

[6] Viewing a rigid body as a collection of points, the distance constraints between the points, as we saw earlier, can be viewed as holonomic constraints.

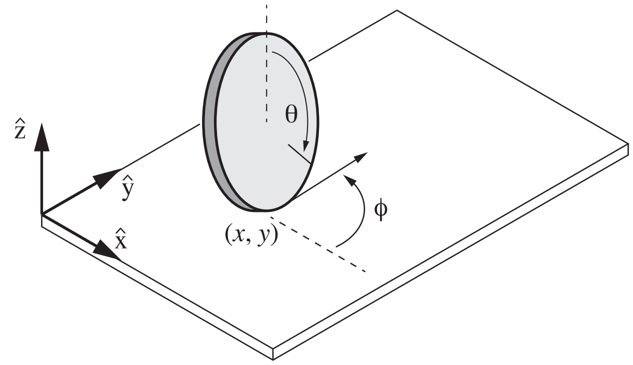


Figure 2.11: A coin rolling on a plane without slipping.

We now express, in mathematical form, the fact that the coin rolls without slipping. The coin must always roll in the direction indicated by $(\cos\phi, \sin\phi)$, with forward speed $r\dot{\theta}$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = r\dot{\theta} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}. \tag{2.9}$$

Collecting the four C-space coordinates into a single vector $q = (q_1, q_2, q_3, q_4) = (x, y, \phi, \theta) \in \mathbb{R}^2 \times T^2$, the above no-slip rolling constraint can then be expressed in the form

$$\begin{bmatrix} 1 & 0 & 0 & -r\cos q_3 \\ 0 & 1 & 0 & -r\sin q_3 \end{bmatrix} \dot{q} = 0. \tag{2.10}$$

These are Pfaffian constraints of the form $A(q)\dot{q} = 0$, $A(q) \in \mathbb{R}^{2\times 4}$.

These constraints are not integrable; that is, for the $A(q)$ given in (2.10), there does not exist any differentiable $g : \mathbb{R}^4 \to \mathbb{R}^2$ such that $\frac{\partial g}{\partial q} = A(q)$. To see why, there would have to exist a differentiable $g_1(q)$ that satisfied the following four equalities:

$$\begin{array}{lclcl} \frac{\partial g_1}{\partial q_1} & = & 1 & \longrightarrow & g_1(q) = q_1 + h_1(q_2, q_3, q_4) \\ \frac{\partial g_1}{\partial q_2} & = & 0 & \longrightarrow & g_1(q) = h_2(q_1, q_3, q_4) \\ \frac{\partial g_1}{\partial q_3} & = & 0 & \longrightarrow & g_1(q) = h_3(q_1, q_2, q_4) \\ \frac{\partial g_1}{\partial q_4} & = & -r\cos q_3 & \longrightarrow & g_1(q) = -rq_4\cos q_3 + h_4(q_1, q_2, q_3), \end{array}$$

for some h_i, $i = 1, \ldots, 4$, differentiable in each of its variables. By inspection it should be clear that no such $g_1(q)$ exists. Similarly, it can be shown that $g_2(q)$ does not exist, so that the constraint (2.10) is nonintegrable. A Pfaffian constraint that is nonintegrable is called a **nonholonomic constraint.** Such constraints reduce the dimension of the feasible velocities of the system, but do not reduce the dimension of the reachable C-space. The rolling coin can reach any point in its four-dimensional C-space despite the two constraints on its velocity.[7] See Exercise 28.

[7]Some texts define the number of degrees of freedom of a system to be the dimension of

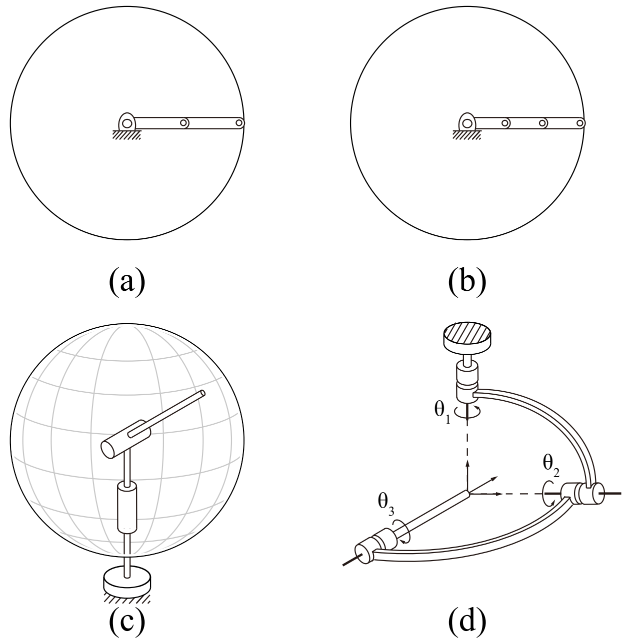


Figure 2.12: Examples of workspaces for various robots: (a) a planar 2R open chain; (b) a planar 3R open chain; (c) a spherical 2R open chain; (d) a 3R orienting mechanism.

Nonholonomic constraints arise in a number of robotics contexts that involve conservation of momentum and rolling without slipping, e.g., wheeled vehicle kinematics and grasp contact kinematics. We examine nonholonomic constraints in greater detail in the later chapter on wheeled robots.

2.5 Task Space and Workspace

We introduce two more concepts relating to the configuration of a robot: the task space and the workspace. Both relate to the configuration of the end-effector of a robot, not the configuration of the entire robot.

The **task space** is a space in which the robot's task can be naturally expressed. For example, if the robot's task is to plot with a pen on a piece of paper, the task space would be $\mathbb{R}^2$. If the task is to manipulate a rigid body, a natural representation of the task space is the C-space of a rigid body, representing the position and orientation of a frame attached to the robot's end-effector. This is

the feasible velocities, e.g., two for the rolling coin. We uniformly refer to the dimension of the C-space as the number of degrees of freedom.

the default representation of task space. The decision of how to define the task space is driven by the task, independent of the robot.

The **workspace** is a specification of the configurations the end-effector of the robot can reach. The definition of the workspace is primarily driven by the robot's structure, independent of the task.

Both the task space and the workspace involve a choice by the user; in particular, the user may decide that some freedoms of the end-effector (e.g., its orientation) do not need to be represented.

The task space and the workspace are distinct from the robot's C-space. A point in the task space or the workspace may be achievable by more than one robot configuration, meaning that the point is not a full specification of the robot's configuration. For example, for an open-chain robot with seven joints, the six-dof position and orientation of its end-effector does not fully specify the robot's configuration.

Some points in task space may not be reachable at all by the robot, e.g., a point on a chalkboard that the robot cannot reach. By definition, all points in the workspace are reachable by at least one configuration of the robot.

Two mechanisms with different C-spaces may have the same workspace. For example, considering the end-effector to be the Cartesian tip of the robot (e.g., the location of a plotting pen) and ignoring orientations, the planar 2R open chain with links of equal length three (Figure 2.12(a)) and the planar 3R open chain with links of equal length two (Figure 2.12(b)) have the same workspace despite having different C-spaces.

Two mechanisms with the same C-space may also have different workspaces. For example, taking the end-effector to be the Cartesian tip of the robot and ignoring orientations, the 2R open chain of Figure 2.12(a) has a planar disk as its workspace, while the 2R open chain of Figure 2.12(c) has the surface of a sphere as its workspace.

Attaching a coordinate frame to the tip of the tool of the 3R open chain "wrist" mechanism of Figure 2.12(d), we see that the frame can achieve any orientation by rotating the joints, but the Cartesian position of the tip is always fixed. This can be seen by noting that the three joint axes always intersect at the tip. For this mechanism, we would likely define the workspace to be the three-dof space of orientations of the frame, $S^2 \times S^1$, which is different from the C-space T^3. The task space depends on the task; if the job is to point a laser pointer, then rotations about the axis of the laser beam are immaterial, and the task space would be S^2, the set of directions the laser can point.

Example 2.7. The SCARA robot of Figure 2.13 is an RRRP open chain that is widely used for tabletop pick-and-place tasks. The end-effector configuration is completely described by the four parameters (x, y, z, ϕ), where (x, y, z) denotes the Cartesian position of the end-effector center point, and ϕ denotes the orientation of the end-effector in the x-y plane. Its task space would typically be defined as $\mathbb{R}^3 \times S^1$, and its workspace would typically be defined as the reachable points in (x, y, z) Cartesian space, since all orientations $\phi \in S^1$ can be achieved at all reachable points.

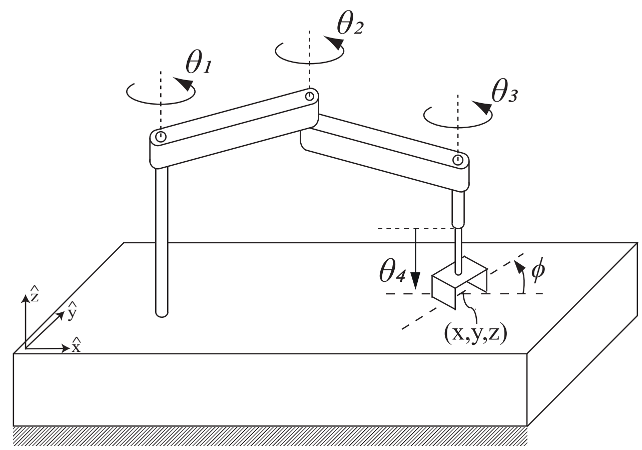


Figure 2.13: SCARA robot.

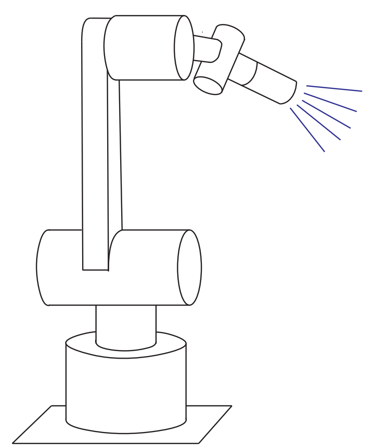


Figure 2.14: A spray-painting robot.

Example 2.8. A standard 6R industrial manipulator can be adapted to spray-painting applications as shown in Figure 2.14. The paint spray nozzle attached to the tip can be regarded as the end-effector. What is important to the task is the Cartesian position of the spray nozzle, together with the direction in which the spray nozzle is pointing; rotations about the nozzle axis (which points in the direction in which paint is being sprayed) do not matter. The nozzle configuration can therefore be described by five coordinates: (x, y, z) for the

Cartesian position of the nozzle and spherical coordinates (θ, ϕ) to describe the direction in which the nozzle is pointing. The task space can be written as $\mathbb{R}^3 \times S^2$. The workspace could be the reachable points in $\mathbb{R}^3 \times S^2$, or, to simplify visualization, the user could define the workspace to be the subset of $\mathbb{R}^3$ corresponding to the reachable Cartesian positions of the nozzle.

2.6 Summary

- A robot is mechanically constructed from **links** that are connected by various types of **joints**. The links are usually modeled as rigid bodies. An **end-effector** such as a gripper may be attached to some link of the robot. **Actuators** deliver forces and torques to the joints, thereby causing motion of the robot.

- The most widely used one-dof joints are the **revolute joint**, which allows for rotation about the joint axis, and the **prismatic joint**, which allows for translation in the direction of the joint axis. Some common two-dof joints include the **cylindrical joint**, which is constructed by serially connecting a revolute and prismatic joint, and the **universal joint**, which is constructed by orthogonally connecting two revolute joints. The **spherical joint**, also known as **ball-in-socket joint**, is a three-dof joint whose function is similar to the human shoulder joint.

- The **configuration** of a rigid body is a specification of the location of all of its points. For a rigid body moving in the plane, three independent parameters are needed to specify the configuration. For a rigid body moving in three-dimensional space, six independent parameters are needed to specify the configuration.

- The configuration of a robot is a specification of the configuration of all of its links. The robot's **configuration space** is the set of all possible robot configurations. The dimension of the C-space is the number of **degrees of freedom** of a robot.

- The number of degrees of freedom of a robot can be calculated using **Grübler's formula**,

  $$\mathrm{dof} = m(N - 1 - J) + \sum_{i=1}^{J} f_i,$$

  where $m = 3$ for planar mechanisms and $m = 6$ for spatial mechanisms, N is the number of links (including the ground link), J is the number of joints, and f_i is the number of degrees of freedom of joint i.

- A robot's C-space can be parametrized explicitly or represented implicitly. For a robot with n degrees of freedom, an **explicit parametrization** uses n coordinates, the minimum necessary. An **implicit representation** involves m coordinates with $m \geq n$, with the m coordinates subject to $m-n$ constraint equations. With the implicit parametrization, a robot's C-space can be viewed as a surface of dimension n embedded in a space of higher dimension m.

- The C-space of an n-dof robot whose structure contains one or more closed loops can be implicitly represented using k **loop-closure equations** of the form $g(\theta) = 0$, where $\theta \in \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^k$. Such constraint equations are called **holonomic constraints**. Assuming $\theta(t)$ varies with time t, the holonomic constraints $g(\theta(t)) = 0$ can be differentiated with respect to t to yield
$$\frac{\partial g}{\partial \theta}(\theta)\dot{\theta} = 0,$$
where $\frac{\partial g}{\partial \theta}(\theta)$ is a $k \times m$ matrix.

- A robot's motion can also be subject to velocity constraints of the form
$$A(\theta)\dot{\theta} = 0,$$
where $A(\theta)$ is a $k \times m$ matrix that cannot be expressed as the differential of some function $g(\theta)$, i.e., there does not exist any $g(\theta), g : \mathbb{R}^m \to \mathbb{R}^k$, such that
$$A(\theta) = \frac{\partial g}{\partial \theta}(\theta).$$
Such constraints are said to be **nonholonomic constraints**, or **nonintegrable constraints**. These constraints reduce the dimension of feasible velocities of the system but do not reduce the dimension of the reachable C-space. Nonholonomic constraints arise in robot systems subject to conservation of momentum or rolling without slipping.

- A robot's **task space** is a space in which the robot's task can be naturally expressed. A robot's **workspace** is a specification of the configurations the end-effector of the robot can reach.

Notes and References

In the kinematics literature, structures that consist of links connected by joints are also called **mechanisms** or **linkages**. The degrees of freedom of mechanisms is treated in most texts on mechanism analysis and design, e.g., [29]. A robot's configuration space has the mathematical structure of a differentiable manifold. Some accessible introductions to differential manifolds and differential geometry are [88], [18]. Configuration spaces are further examined in a motion planning context in [59], [19].

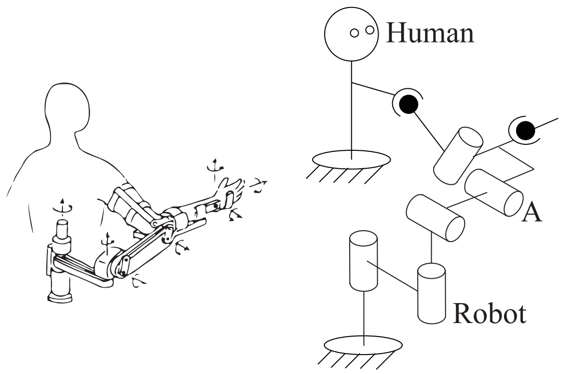


Figure 2.15: Robot used for human arm rehabilitation.

2.7 Exercises

1. Using the methods of Section 2.1, derive a formula, in terms of n, for the degrees of freedom of a rigid body in n-dimensional space. Indicate how many of those dof are translational and how many are rotational. Describe the topology of the C-space (e.g., for $n = 2$, the topology is $\mathbb{R}^2 \times S^1$).

2. Find the degrees of freedom of your arm, from your torso to your palm (just past the wrist, not including finger degrees of freedom). Do this in two ways: (a) add up the degrees of freedom at the shoulder, elbow, and wrist joints, and (b) fix your palm flat on a table with your elbow bent, and without moving your torso, investigate how many degrees of freedom with which you can still move your arm. Do your answers agree? How many constraints were placed on your arm when you placed your palm at a fixed configuration on the table?

3. Assume each of your arms has n degrees of freedom. You are driving a car, your torso is stationary relative to the car (a tight seatbelt!), and both hands are firmly grasping the wheel, so that your hands do not move relative to the wheel. How many degrees of freedom does your arms-plus-steering wheel system have? Explain your answer.

4. Figure 2.15 shows a robot used for human arm rehabilitation. Determine the degrees of freedom of the chain formed by the human arm and robot.

5. The mobile manipulator of Figure 2.16 consists of a 6R arm and multifingered hand mounted on a mobile base with a single wheel. The wheel rotates

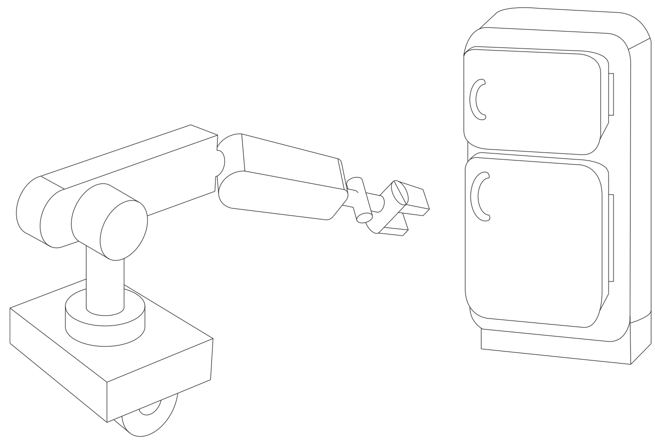


Figure 2.16: Mobile manipulator.

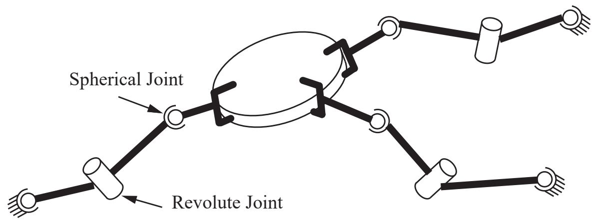


Figure 2.17: Three cooperating SRS arms grasping a common object.

without slip, and its axis of rotation always remains parallel to the ground.
(a) Ignoring the multifingered hand, describe the configuration space of the mobile manipulator.
(b) Now suppose the robot hand rigidly grasps the refrigerator door handle, and with its wheel completely stationary, opens the door using only its arm. With the door open, how many degrees of freedom does the mechanism formed by the arm and open door have?
(c) A second identical mobile manipulator comes along, and after parking its mobile base, also rigidly grasps the refrigerator door handle. How many degrees of freedom does the mechanism formed by the two arms and the open referigerator door have?

6. Three identical SRS open chain arms are grasping a common object as shown in Figure 2.17.
(a) Find the degrees of freedom of this system.
(b) Suppose there are now a total of n such arms grasping the object. What is the degrees of freedom of this system?

(c) Suppose the spherical wrist joint in each of the n arms is now replaced by a universal joint. What is the degrees of freedom of the overall system?

7. Consider a spatial parallel mechanism consisting of a moving plate connected to the fixed plate by n identical open chain legs. In order for the moving plate to have six degrees of freedom, how many degrees of freedom should each leg have, as a function of n? For example, if $n = 3$, then the moving plate and fixed plate are connected by three open chains; how many degrees of freedom should each open chain have in order for the moving plate to move with six degrees of freedom? Solve for arbitrary n.

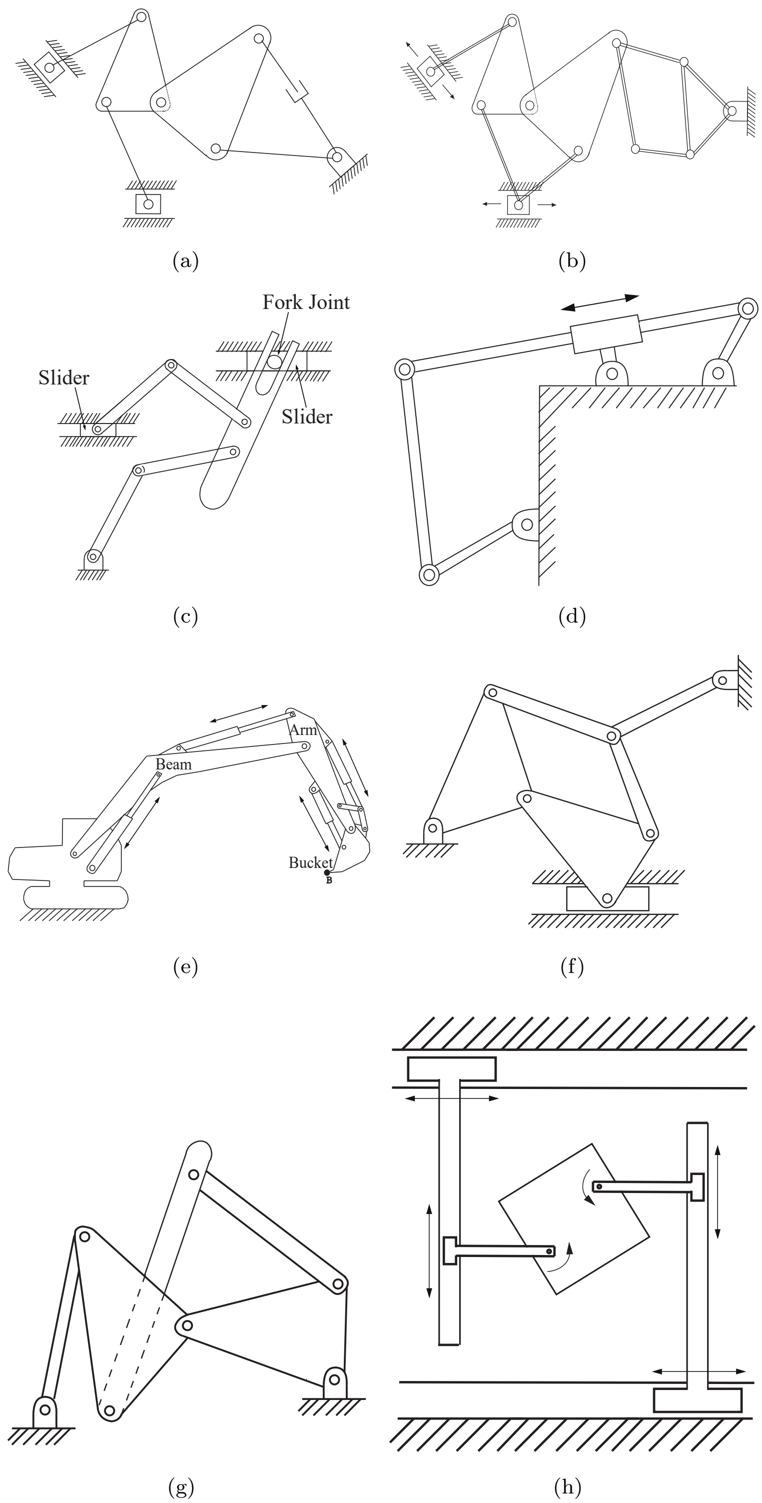
8. Use the planar version of Grübler's formula to determine the degrees of freedom of the mechanisms shown in Figure 2.18. Comment on whether your results agree with your intuition about the possible motions of these mechanisms.

9. Use the spatial version of Grübler's formula to determine the degrees of freedom of the mechanisms shown in Figure 2.19. Comment on whether your results agree with your intuition about the possible motions of these mechanisms.

10. In the parallel mechanism shown in Figure 2.20, six legs of identical length are connected to a fixed and moving platform via spherical joints. Determine the degrees of freedom of this mechanism using Grübler's formula. Illustrate all possible motions of the upper platform.

11. The $3 \times$ UPU platform of Figure 2.21 consists of two platforms–the lower one stationary, the upper one mobile–connected by three UPU serial chains.
(a) Using the spatial version of Grübler's formula, Verify that it has three degrees of freedom.
(b) Construct a physical model of the $3 \times$ UPU platform to see if it indeed has three degrees of freedom. In particular, lock the three P joints in place; does the robot become a structure as predicted by Grübler's formula, or does it move? Try reversing the order of the U joints, i.e., with the rotational axes connecting the leg to the platforms arranged parallel to the platforms, and also arranged orthogonal to the platforms. Does the order in which the U joints are connected matter?

12. (a) The mechanism of Figure 2.22(a) consists of six identical squares arranged in a single closed loop, connected serially by revolute joints. The bottom square is fixed to ground. Determine its degrees of freedom using an appropriate version of Grübler's formula.
(b) The mechanism of Figure 2.22(b) also consists of six identical squares connected by revolute joints, but arranged differently as shown. Determine its degrees of freedom using an appropriate version of Grübler's formula. Does your result agree with your intuition about the possible motions of this mechanism?


(a)
(b)
Fork Joint
Slider
Slider
(c)
(d)
Arm
Beam
Bucket
B
(e)
(f)
(g)
(h)

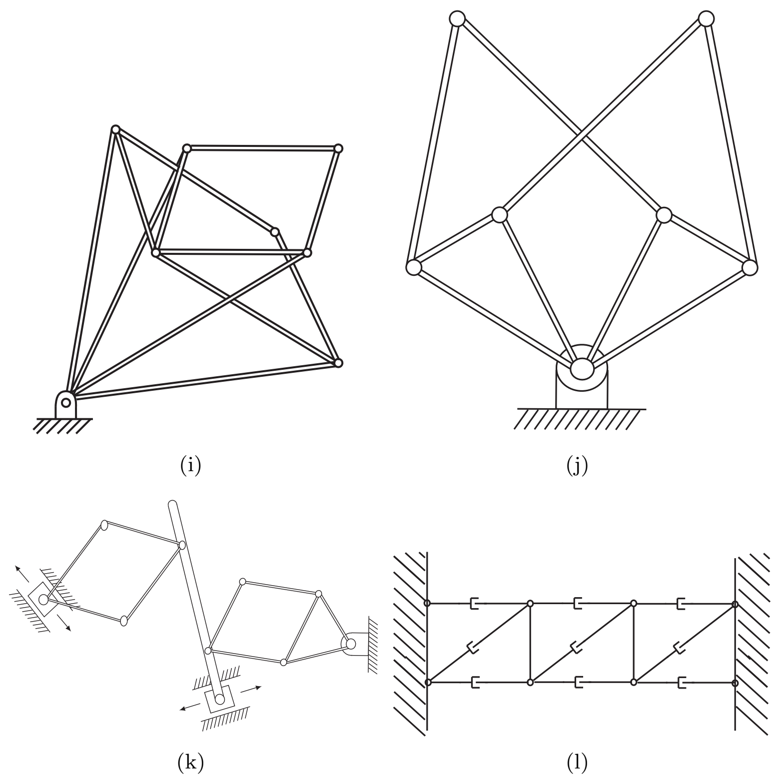
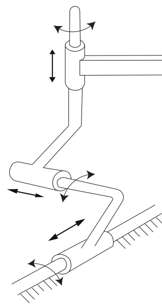


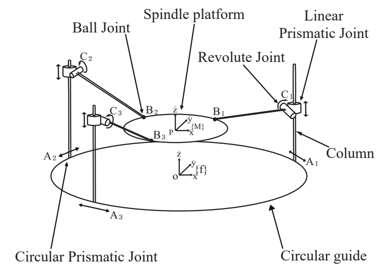
Figure 2.18: A collection of planar mechanisms.

13. Figure 2.23 shows a spherical four-bar linkage, in which four links (one of the links is the ground link) are connected by four revolute joints to form a single-loop closed chain. The four revolute joints are arranged so that they lie on a sphere, and such that their joint axes intersect at a common point.
(a) Use an appropriate version of Grübler's formula to find the degrees of freedom. Justify your choice of formula.
(b) Describe the configuration space.
(c) Assuming a reference frame is attached to the center link, describe its workspace.

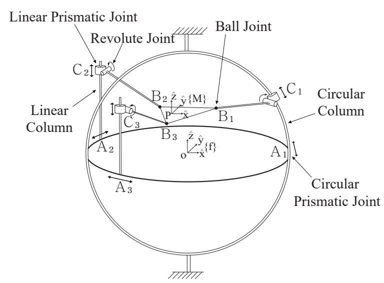
14. Figure 2.24 shows a parallel robot for surgical applications. As shown in Figure 2.24(a), leg A is an RRRP chain, while legs B and C are RRRUR chains. A surgical tool is attached to the end-effector as shown.
(a) Use an appropriate version of Grübler's formula to find the degrees of freedom of the mechanism in Figure 2.24(a).



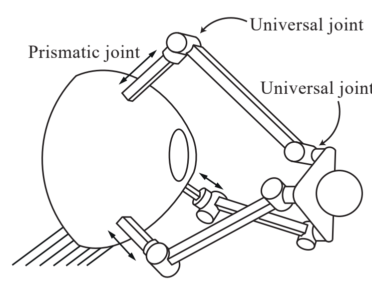
(a)


Ball Joint
Spindle platform
Linear
Prismatic Joint
Revolute Joint
C_2
C_1
C_3
B_2
B_1
B_3
{M}
{f}
A_2
A_1
A_3
Column
Circular Prismatic Joint
Circular guide

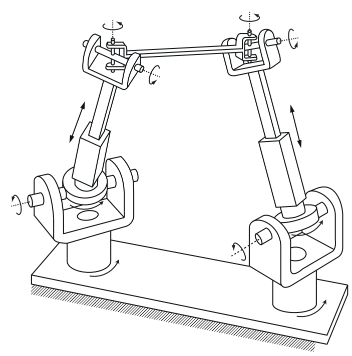
(b)


Linear Prismatic Joint
Revolute Joint
Ball Joint
C_2
C_1
C_3
B_2
B_1
B_3
{M}
{f}
Linear
Column
Circular
Column
A_2
A_1
A_3
Circular
Prismatic Joint

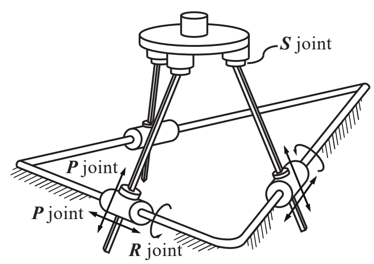
(c)


Universal joint
Prismatic joint
Universal joint

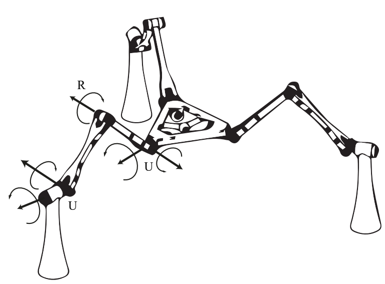
(d)



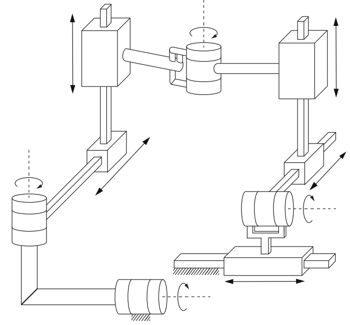
(e)


S joint
P joint
P joint
R joint

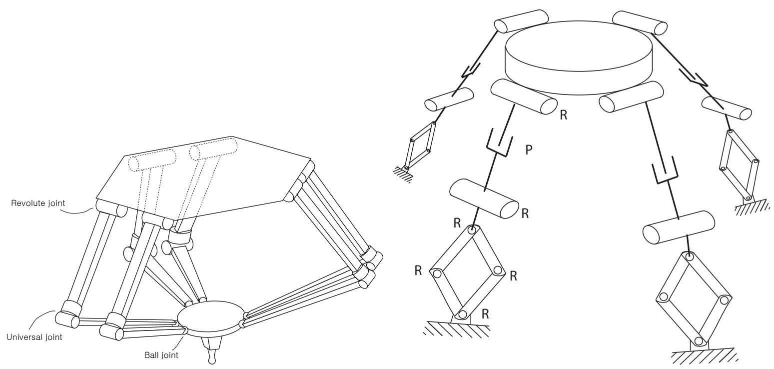
(f)


R
U
U

(g)



(h)



(i) (j)

Figure 2.19: A collection of spatial parallel mechanisms.

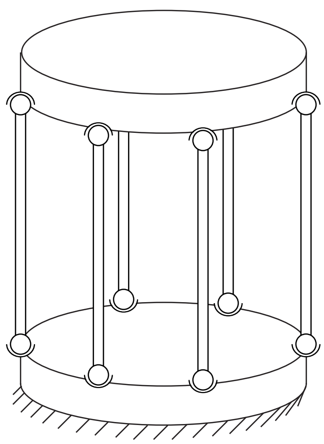


Figure 2.20: A $6 \times SS$ platform.

(b) Now suppose the surgical tool must always pass through point A in Figure 2.24(a). How many degrees of freedom does the manipulator have?
(c) Legs A, B, and C are now replaced by three identical RRRR legs as shown in Figure 2.24(b). Furthermore the axes of all R joints pass through point A. Use an appropriate version of Grübler's formula to derive the degrees of freedom of this mechanism.

15. Figure 2.25 shows a $3 \times$ PUP platform, in which three identical PUP legs connect a fixed base to a moving platform. As shown in the figure, the P joints on both the fixed base and moving platform are arranged symmetrically. Recalling that the U joint consists of two revolute joints connected orthogonally,

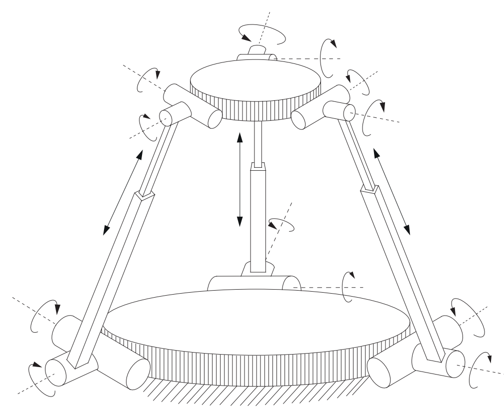


Figure 2.21: The $3 \times$ UPU platform.

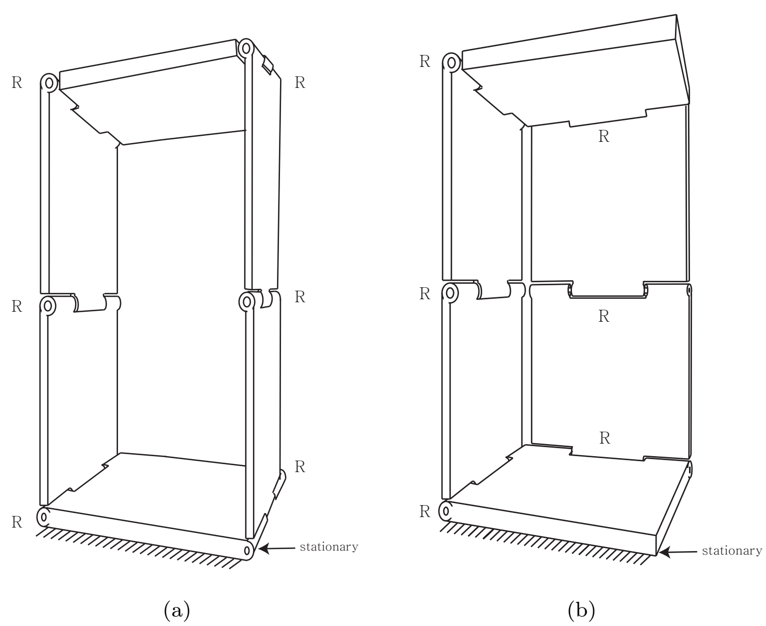


Figure 2.22: Two mechanisms.

each R joint connected to the moving platform has its joint axis aligned in the

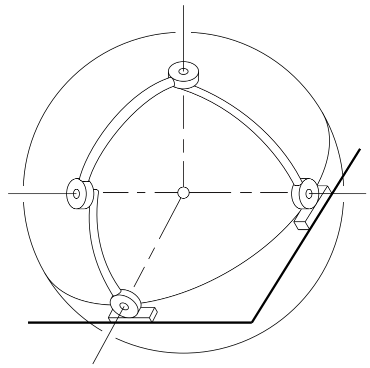
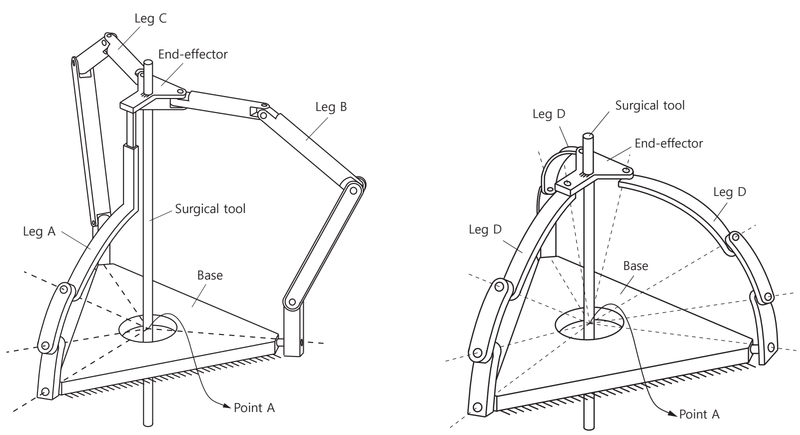


Figure 2.23: The spherical four-bar linkage.



(a) (b)

Figure 2.24: Surgical manipulator.

same direction as the platform's P joint. The R joint connected to the fixed base has its joint axis orthogonal to the base P joint. Use an appropriate version of Grübler's formula to find the degrees of freedom. Does your answer agree with your intuition about this mechanism? If not, try to explain any discrepancies without resorting to a detailed kinematic analysis.

16. The dual-arm robot of Figure 2.26 is rigidly grasping a box as shown. The box can only slide on the table; the bottom face of the box must always be in contact with the table. How many degrees of freedom does this system have?

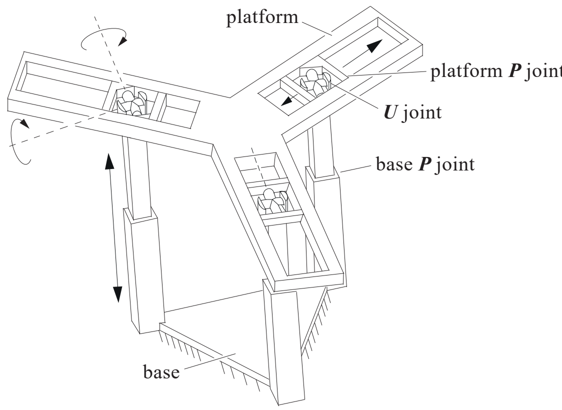


Figure 2.25: The $3 \times PUP$ platform.

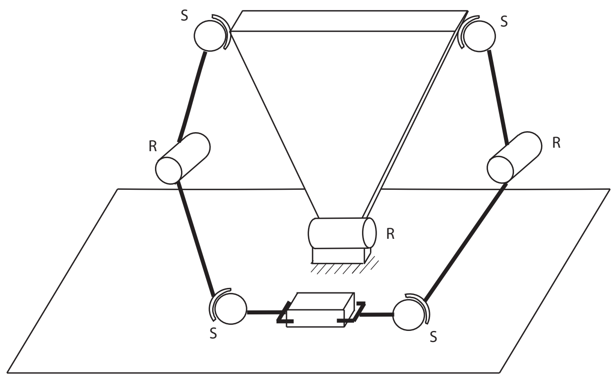


Figure 2.26: Dual arm robot.

17. The dragonfly robot of Figure 2.27 has a body, four legs, and four wings as shown. Each leg is connected to its adjacent leg by a USP open chain. Use appropriate versions of Grübler's formula to answer the following questions:
(a) Suppose the body is fixed, and only the legs and wings can move. How many degrees of freedom does the robot have?
(b) Now suppose the robot is flying in the air. How many degrees of freedom does the robot have?
(c) Now suppose the robot is standing with all four feet in contact with the ground. Assume the ground is uneven, and that each foot-ground contact can be modeled as a point contact with no slip. How many degrees of freedom does the robot have?

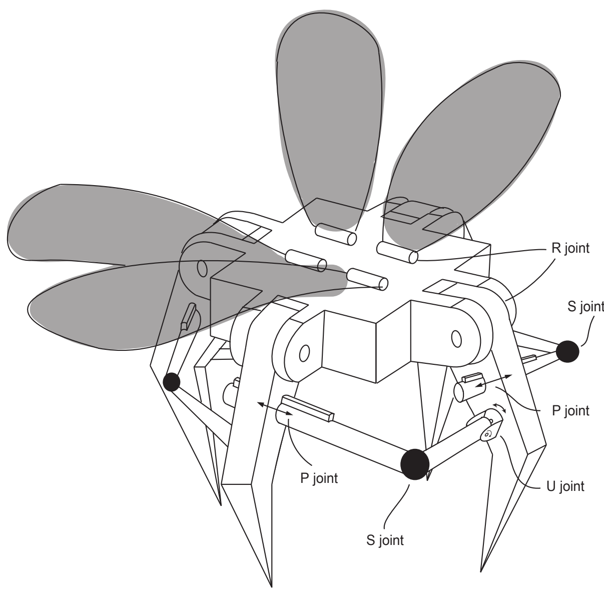
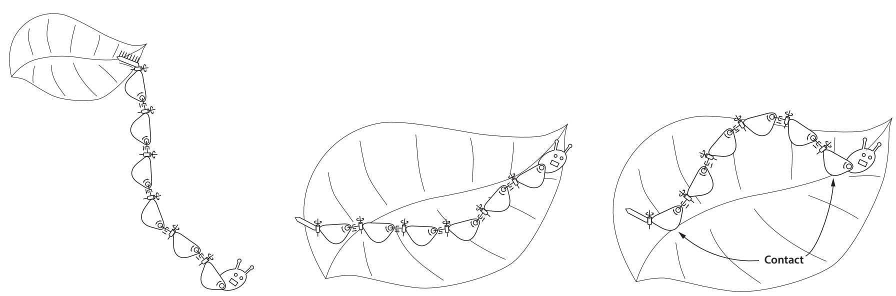


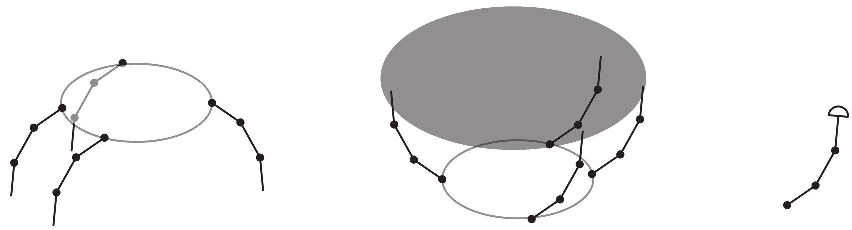
Figure 2.27: Dragonfly robot.



(a) (b) (c)

Figure 2.28: A caterpillar robot.

18. (a) A caterpillar robot is hanging by its tail as shown in Figure 2.28(a). The caterpillar robot consists of eight rigid links (one head, one tail, and six body links) connected serially by revolute-prismatic pairs as shown. Find the degrees of freedom of this robot.
(b) The caterpillar robot is now crawling on a leaf as shown in Figure 2.28(b). Suppose all six body links must be in contact with the leaf at all times (each link-leaf contact can be modelled as a frictionless point contact). Find the de-



(a) (b) (c)

Figure 2.29: (a) A four-fingered hand with palm. (b) The hand grasping an ellipsoidal object. (c) A rounded fingertip that can roll on the object without sliding.

grees of freedom of this robot during crawling.
(c) Now suppose the caterpillar robot crawls on the leaf as shown in Figure 2.28(c), in which only the first and last body links are in contact with the leaf. Find the degrees of freedom of this robot during crawling.

19. The four-fingered hand of Figure 2.29(a) consists of a palm and four URR fingers (the U joints connect the fingers to the palm).
(a) Assume the fingertips are points, and that one of the fingertips is in contact with the table surface (sliding of the fingertip point contact is allowed). How many degrees of freedom does the hand have? WHat if two fingertips are in sliding point contact with the table? Three? All four?
(b) Repeat part (a) but with each URR finger replaced by an SRR finger (each universal joint is replaced by a spherical joint).
(c) The hand (with URR fingers) now grasps an ellipsoidal object as shown in Figure 2.29(b). Assume the palm is fixed in space, and that no slip occurs between the fingertips and object. How many degrees of freedom does the system have?
(d) Now assume the fingertips are spheres as shown in Figure 2.29(c). Each of the fingertips can roll on the object, but cannot slip or break contact with the object. How many degrees of freedom does the system have? For a single fingertip in rolling contact with theobject, comment on the dimension of the space of feasible fingertip velocities relative to the object versus the number of parameters needed to represent the fingertip configuration relative to the object (its degrees of freedom). (Hint: You may want to experiment by rolling a ball around on a tabletop to get some intuition.)

20. Consider the slider-crank mechanism of Figure 2.4(b). A rotational motion at the revolute joint fixed to ground (the "crank") causes a translational motion at the prismatic joint (the "slider"). Suppose the two links connected to the crank and slider are of equal length. Determine the configuration space of this mechanism, and draw its projected version on the space defined by the crank

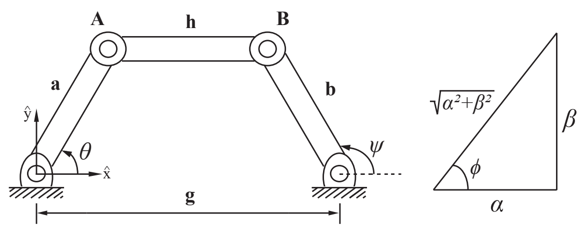


Figure 2.30: Planar four-bar linkage.

and slider joint variables.

21. (a) Use an appropriate version of Grübler's formula to determine the degrees of freedom of a planar four-bar linkage floating in space.
(b) Derive an implicit parametrization of the four-bar's configuration space as follows. First, label the four links 1,2,3,4, and choose three points A, B, C on link 1, D, E, F on link 2, G, H, I on link 3, and J, K, L on link 4. The four-bar linkage is constructed such that the four following pairs of points are each connected by a revolute joint: C with D, F with G, I with J, and L with A. Write down explicit constraints on the coordinates for the eight points $A, \ldots H$ (assume a fixed reference frame has been chosen, and denote the coordinates for point A by $p_A = (x_A, y_A, z_A)$, and similarly for the other points). Based on counting the number of variables and constraints, what is the degrees of freedom of the configuration space? If it differs from the result you obtained in (a), try to explain why.

22. In this exercise we examine in more detail the representation of the C-space for the planar four-bar linkage of Figure 2.30. Attach a fixed reference frame and label the joints and link lengths as shown in the figure. The (x, y) coordinates for joints A and B are given by

$$\begin{aligned} A(\theta) &= (a\cos\theta,\ a\sin\theta)^T \\ B(\psi) &= (g + b\cos\psi,\ b\sin\psi)^T. \end{aligned}$$

Using the fact that the link connecting A and B is of fixed length h, i.e., $\|A(\theta) - B(\psi)\|^2 = h^2$, we have the constraint

$$b^2 + g^2 + 2gb\cos\psi + a^2 - 2(a\cos\theta(g + b\cos\psi) + ab\sin\theta\sin\psi) = h^2.$$

Grouping the coefficients of $\cos\psi$ and $\sin\psi$, the above equation can be expressed in the form

$$\alpha(\theta)\cos\psi + \beta(\theta)\sin\psi = \gamma(\theta), \tag{2.11}$$

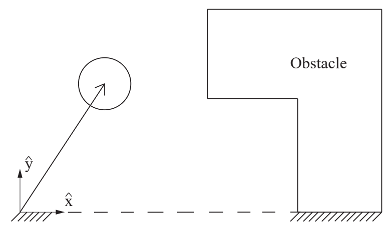


Figure 2.31: A circular disc robot moving in the plane.

where

$$\begin{aligned}\alpha(\theta) &= 2gb - 2ab\cos\theta \\ \beta(\theta) &= -2ab\sin\theta \\ \gamma(\theta) &= h^2 - g^2 - b^2 - a^2 + 2ag\cos\theta.\end{aligned}$$

We now express ψ as a function of θ, by first dividing both sides of Equation (2.11) by $\sqrt{\alpha^2+\beta^2}$ to obtain

$$\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}\cos\psi + \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\sin\psi = \frac{\gamma}{\sqrt{\alpha^2+B^2}}$$

Referring to Figure 2.30(b), the angle ϕ is given by $\phi = \tan^{-1}(\beta/\alpha)$, so that Equation (22) becomes

$$\cos(\psi - \phi) = \frac{\gamma}{\sqrt{\alpha^2+\beta^2}}.$$

Therefore,

$$\psi = \tan^{-1}(\frac{\beta}{\alpha}) \pm \cos^{-1}\left(\frac{\gamma}{\sqrt{\alpha^2+\beta^2}}\right).$$

(a) Note that a solution exists only if $\gamma^2 \leq \alpha^2 + \beta^2$. What are the physical implications if this constraint is not satisfied?
(b) Note that for each value of input angle θ, there exists two possible values of the output angle ψ. What do these two solutions look like?
(c) Draw the configuration space of the mechanism in θ-ψ space for the following link length values: $a = b = g = h = 1$.
(d) Repeat (c) for the following link length values: $a = 1$, $b = 2$, $h = \sqrt{5}$, $g = 2$.
(e) Repeat (c) for the following link length values: $a = 1$, $b = 1$, $h = 1$, $g = \sqrt{3}$.

23. A circular disc robot moves in the plane as shown in Figure 2.31. An L-shaped obstacle is nearby. Using the (x, y) coordinates of the robot's center as its configuration space coordinates, draw the free configuration space (i.e., the set of all feasible configurations) of the robot.

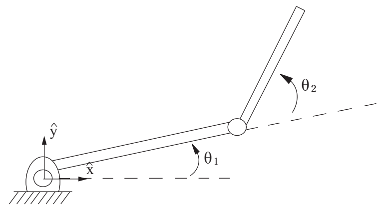


Figure 2.32: Two-link planar 2R open chain.

24. The tip coordinates for the two-link planar 2R robot of Figure 2.32 are given by

$$x = 2\cos\theta_1 + \cos(\theta_1 + \theta_2)$$
$$y = 2\sin\theta_1 + \sin(\theta_1 + \theta_2).$$

(a) What is the robot's configuration space?
(b) What is the robot's task space (i.e., the set of all points reachable by the tip)?
(c) Suppose infinitely long vertical barriers are placed at $x = 1$ and $x = -1$. What is the free C-space of the robot (i.e., the portion of the C-space that does not result in any collisions with the vertical barriers)?

25. (a) Consider a planar 3R open chain with link lengths (starting from the fixed base joint) 5, 2, and 1, respectively. Considering only the Cartesian point of the tip, draw its workspace.
(b) Now consider a planar 3R open chain with link lengths (starting from the fixed base joint) 1, 2, and 5, respectively. Considering only the Cartesian point of the tip, draw its workspace. Which of these two chains has a larger workspace?
(c) A not-so-clever designer claims that he can make the workspace of any planar open chain larger simply by increasing the length of the last link. Explain the fallacy behind this claim.

26. (a) Describe the task space for a robot arm writing on a blackboard.
(b) Describe the task space for a robot arm twirling a baton.

27. Give a mathematical description of the topologies of the C-spaces of the following systems. Use cross-products, as appropriate, of spaces such as a closed interval of a line $[a, b]$ and $\mathbb{R}^k$, S^m, and T^n, where k, m, and n are chosen appropriately.

(i) The chassis of a car-like mobile robot rolling on an infinite plane.

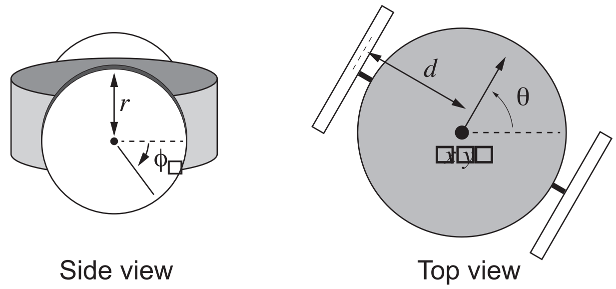


Figure 2.33: A side view and a top view of a diff-drive robot.

(ii) The car-like mobile robot, but including a representation of the wheel configurations.

(iii) The car-like mobile robot driving around on a spherical asteroid.

(iv) The car-like mobile robot on an infinite plane with an RRPR robot arm mounted on it. The prismatic joint has joint limits, but the revolute joints do not.

(v) A free-flying spacecraft with a 6R arm mounted on it, no joint limits.

28. Describe an algorithm that drives the rolling coin of Figure 2.11 from any arbitrary initial configuration in its four-dimensional C-space to any arbitrary goal configuration, despite the two nonholonomic constraints. The control inputs are the rolling speed $\dot{\theta}$ and the turning speed $\dot{\phi}$.

29. A differential-drive mobile robot has two wheels which do not steer but whose speeds can be controlled independently. The robot goes forward and backward by spinning the wheels in the same direction at the same speed, and it turns by spinning the wheels at different speeds. The configuration of the robot is given by five variables: the (x, y) location of the point halfway between the wheels, the heading direction θ of the robot's chassis relative to the x-axis of the world frame, and the rotation angles ϕ_1 and ϕ_2 of the two wheels about the axis through the centers of the wheels (Figure 2.33). Assume that the radius of each wheel is r and the distance between the wheels is $2d$.

(i) Let $q = (x, y, \theta, \phi_1, \phi_2)^T$ be the configuration of the robot. If the two control inputs are the angular velocities of the wheels $\omega_1 = \dot{\phi}_1$ and $\omega_2 = \dot{\phi}_2$, write the vector differential equation $\dot{q} = g_1(q)\omega_1 + g_2(q)\omega_2$. The vector fields $g_1(q)$ and $g_2(q)$ are called *control vector fields*, expressing how the system moves when the respective control is applied.

(ii) Write the corresponding Pfaffian constraints $A(q)\dot{q} = 0$ for this system. How many Pfaffian constraints are there?

(iii) Are the constraints holonomic or nonholonomic? Or how many are holonomic and how many nonholonomic?

30. Determine if the following differential constraints are holonomic or nonholonomic:

(a)

$$(1 + \cos q_1)\dot{q}_1 + (1 + \cos q_2)\dot{q}_2 + (\cos q_1 + \cos q_2 + 4)\dot{q}_3 = 0.$$

(b)

$$\begin{aligned} -\dot{q}_1 \cos q_2 + \dot{q}_3 \sin(q_1 + q_2) - \dot{q}_4 \cos(q_1 + q_2) &= 0 \\ \dot{q}_3 \sin q_1 - \dot{q}_4 \cos q_1 &= 0. \end{aligned}$$

Chapter 3

Rigid-Body Motions

In the previous chapter, we saw that a minimum of six numbers are needed to specify the position and orientation of a rigid body in three-dimensional physical space. In this chapter we develop a systematic way to describe a rigid body's position and orientation that relies on attaching a reference frame to the body. The configuration of this frame with respect to a fixed reference frame is then represented as a 4×4 matrix. This is an example of an implicit representation of the C-space, as discussed in the previous chapter: the actual six-dimensional space of rigid-body configurations is obtained by applying ten constraints to the sixteen-dimensional space of 4×4 real matrices.

Such a matrix not only represents the configuration of a frame, but it can also be used to (1) translate and rotate a vector or a frame, and (2) change the representation of a vector or a frame from coordinates in one frame (e.g., $\{a\}$) to coordinates in another frame (e.g., $\{b\}$). These operations can be performed by simple linear algebra, which is a major reason we choose to represent a configuration as a 4×4 matrix.

The non-Euclidean (non-"flat") nature of the C-space of positions and orientations leads us to use the matrix representation. A rigid body's velocity, however, can be represented simply as a point in $\mathbb{R}^6$: three angular velocities and three linear velocities, which together we call a *spatial velocity* or *twist*. More generally, even though a robot's C-space may not be Euclidean, the set of feasible velocities at any point in the C-space always forms a Euclidean space. As an example, consider a robot whose C-space is the sphere S^2: although the C-space is not flat, the velocity at any configuration can be represented by two real numbers (an element of $\mathbb{R}^2$), such as the rate of change of the latitude and the longitude. At any point on the sphere, the space of velocities can be thought of as the plane (a Euclidean space) tangent to that point on the sphere.

Any rigid-body configuration can be achieved by starting from the fixed (home) reference frame and integrating a constant twist for a specified time. Such a motion resembles the motion of a screw, rotating about and translating along the same fixed axis. The observation that all configurations can be achieved by a screw motion motivates a six-parameter representation of the configuration called the *exponential coordinates*. The six parameters can be divided into parameters to describe the direction of the screw axis and a scalar to indicate how far the screw motion must be followed to achieve the desired configuration.

This chapter concludes with a discussion of forces. Just as angular and linear velocities are packaged together into a single vector in $\mathbb{R}^6$, moments (torques) and forces are packaged together into a six-vector called a *spatial force* or *wrench*.

To illustrate the concepts and to provide a synopsis of the chapter, we begin with a motivating planar example. Before doing so, we make some remarks about vector notation.

A Word about Vectors and Reference Frames

A **free vector** is a geometric quantity with a length and a direction. Think of it as an arrow in $\mathbb{R}^n$. It is called "free" because it is not necessarily rooted anywhere; only its length and direction matter. A linear velocity can be viewed as a free vector: the length of the arrow is the speed and the direction of the arrow is the direction of the velocity. A free vector is denoted by a regular text symbol, e.g., v.

If a reference frame and length scale have been chosen for the underlying space in which v lies, then this free vector can be moved so the base of the arrow is at the origin, without changing the orientation. The free vector v can then be represented as a column vector in the coordinates of the reference frame. This vector is written in italics, $v \in \mathbb{R}^n$, where v is at the "head" of the arrow in the frame's coordinates. If a different reference frame and length scale are chosen, then the representation v will change, but the underlying free vector v is unchanged.

In other words, we say that v is *coordinate free*; it refers to a physical quantity in the underlying space, and it does not care how we represent it. On the other hand, v is a representation of v that depends on a choice of a coordinate frame.

A point p in physical space can also be represented as a vector. Given a choice of reference frame and length scale for physical space, the point p can be represented as a vector from the reference frame origin to p; its vector representation is denoted in italics by $p \in \mathbb{R}^n$. Here, as before, a different choice of reference frame and length scale for physical space leads to a different representation $p \in \mathbb{R}^n$ for the same point p in physical space. See Figure 3.1.

In the rest of this book, a choice of length scale will always be assumed, but we will be dealing with reference frames at different positions and orientations. A reference frame can be placed anywhere in space, and any reference frame leads to an equally valid representation of the underlying space and objects in it. However, we always assume that exactly one stationary **fixed frame**, or **space frame**, denoted $\{s\}$, has been defined. This might be attached to a corner of a room, for example. Similarly, we often assume that at least one frame has been attached to some moving rigid body, such as the body of a quadrotor flying in the room. The **body frame**, denoted $\{b\}$, is the stationary frame that is coincident with the body-attached frame at any instant.

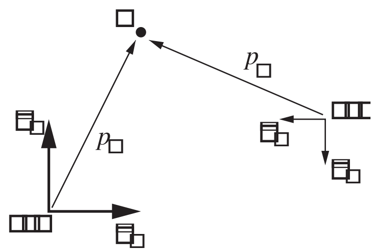


Figure 3.1: The point p exists in physical space, and it does not care how we represent it. If we fix a reference frame {a}, with unit coordinate axes $\hat{x}_a$ and $\hat{y}_a$, we can represent p as $p_a = (1, 2)$. If we fix a reference frame {b} at a different location, a different orientation, and a different length scale, we can represent p as $p_b = (4, -2)$.

Important! All frames in this book are stationary, inertial frames. When we refer to a body frame {b}, we mean a motionless frame that is instantaneously coincident with a frame that is fixed to a (possibly moving) body. This is important to keep in mind, since you may have had a dynamics course that used non-inertial moving frames. Do not confuse these with the stationary, inertial body frames of this book.

For simplicity, we refer to a body frame as a frame attached to a moving rigid body. Despite this, at any instant, by "body frame" we mean the stationary frame that is coincident with the frame moving along with the body.

It is worth repeating to yourself one more time: **all frames are stationary**.

While it is common to attach the origin of the {b} frame to some important point on the body, such as its center of mass, this is not required. The origin of the {b} frame might not even be on the physical body itself, as long as its location relative to the body, viewed from an observer on the body that is stationary relative to the body, is constant.

3.1 Rigid-Body Motions in the Plane

Consider the planar body of Figure 3.2, whose motion is confined to the plane. Suppose a length scale and a fixed reference frame have been chosen as shown. We call the fixed reference frame the fixed frame, or the space frame, denoted {s}, and label its unit axes $\hat{x}_s$ and $\hat{y}_s$. (Throughout this book, the ˆ notation indicates a unit vector.) Similarly, we attach a reference frame with unit axes $\hat{x}_b$ and $\hat{y}_b$ to the planar body. Because this frame moves with the body, it is called the body frame, and is denoted {b}.

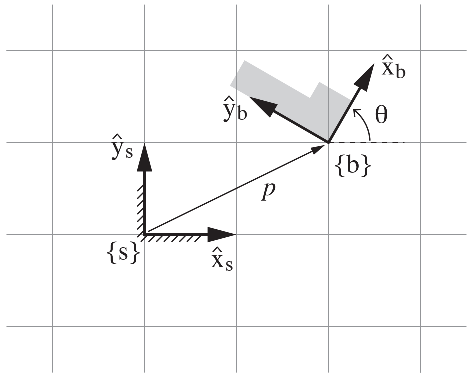


Figure 3.2: The body frame {b} in fixed-frame coordinates {s} is represented by the vector p and the direction of the unit axes $\hat{\mathrm{x}}_\mathrm{b}$ and $\hat{\mathrm{y}}_\mathrm{b}$ expressed in {s}. In this example, $p = (2, 1)^T$ and $\theta = 60°$, so $\hat{\mathrm{x}}_\mathrm{b} = (\cos\theta, \sin\theta)^T = (0.5, 1/\sqrt{2})^T$ and $\hat{\mathrm{y}}_\mathrm{b} = (-\sin\theta, \cos\theta)^T = (-1/\sqrt{2}, 0.5)^T$.

To describe the configuration of the planar body, only the position and orientation of the body frame with respect to the fixed frame needs to be specified. The body frame origin p can be expressed in terms of the coordinate axes of {s} as

$$p = p_x \hat{\mathrm{x}}_\mathrm{s} + p_y \hat{\mathrm{y}}_\mathrm{s}. \tag{3.1}$$

You are probably more accustomed to writing this vector as simply $p = (p_x, p_y)$; this is fine when there is no possibility of ambiguity about reference frames, but writing p as in Equation (3.1) clearly indicates the reference frame with respect to which (p_x, p_y) is defined.

The simplest way to describe the orientation of the body frame {b} relative to the fixed frame {s} is by specifying the angle θ as shown in Figure 3.2. Another (admittedly less simple) way is to specify the directions of the unit axes $\hat{\mathrm{x}}_\mathrm{b}$ and $\hat{\mathrm{y}}_\mathrm{b}$ of {b} relative to {s}, in the form

$$\begin{aligned} \hat{\mathrm{x}}_\mathrm{b} &= \cos\theta\, \hat{\mathrm{x}}_\mathrm{s} + \sin\theta\, \hat{\mathrm{y}}_\mathrm{s} && (3.2) \\ \hat{\mathrm{y}}_\mathrm{b} &= -\sin\theta\, \hat{\mathrm{x}}_\mathrm{s} + \cos\theta\, \hat{\mathrm{y}}_\mathrm{s}. && (3.3) \end{aligned}$$

At first sight this seems a rather inefficient way to represent the body frame orientation. However, imagine the body were to move arbitrarily in three-dimensional space; a single angle θ alone clearly would not suffice to describe the orientation of the displaced reference frame. We would in fact need three angles, but it is not yet clear how to define an appropriate set of three angles. On the other hand, expressing the directions of the coordinate axes of {b} in terms of coefficients of the coordinate axes of {s}, as we have done above for the planar case, is straightforward.

Assuming we agree to express everything in terms of {s}, then just as the

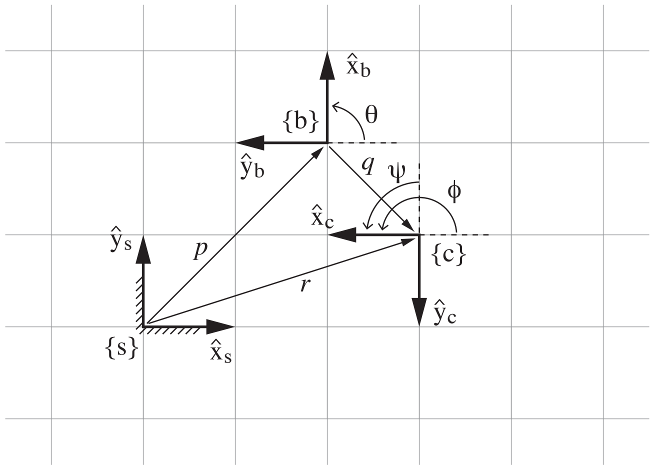


Figure 3.3: The frame $\{b\}$ in $\{s\}$ is given by (P, p), and the frame $\{c\}$ in $\{b\}$ is given by (Q, q). From these we can derive the frame $\{c\}$ in $\{s\}$, described by (R, r). The numerical values of the vectors p, q, and r, and the coordinate axis directions of the three frames, are evident from the grid of unit squares.

point p can be represented as a column vector $p \in \mathbb{R}^2$ of the form

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}, \tag{3.4}$$

the two vectors $\hat{x}_b$ and $\hat{y}_b$ can also be written as column vectors and packaged into the following 2×2 matrix P,

$$P = [\hat{x}_b \;\; \hat{y}_b] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{3.5}$$

The matrix P is an example of a **rotation matrix**. Although P is constructed of four numbers, they are subject to three constraints (each column of P must be a unit vector, and the two columns must be orthogonal to each other), and the one remaining degree of freedom is parametrized by θ. Together, the pair (P, p) provides a description of the orientation and position of $\{b\}$ relative to $\{s\}$.

Now refer to the three frames in Figure 3.3. Repeating the approach above, and expressing $\{c\}$ in $\{s\}$ as the pair (R, r), we can write

$$r = \begin{bmatrix} r_x \\ r_y \end{bmatrix}, \quad R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \tag{3.6}$$

We could also describe the frame $\{c\}$ relative to $\{b\}$. Letting q denote the vector from the origin of $\{b\}$ to the origin of $\{c\}$ expressed in $\{b\}$ coordinates, and letting Q denote the orientation of $\{c\}$ relative to $\{b\}$, we can write $\{c\}$

relative to {b} as the pair (Q, q), where

$$q = \begin{bmatrix} q_x \\ q_y \end{bmatrix}, \quad Q = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}. \tag{3.7}$$

If we know (Q, q) (the configuration of {c} relative to {b}) and (P, p) (the configuration of {b} relative to {s}), we can compute the configuration of {c} relative to {s} as follows:

$$R = PQ \quad \text{(convert } Q \text{ to the \{s\} frame)} \tag{3.8}$$

$$r = Pq + p \quad \text{(convert } q \text{ to the \{s\} frame and vector sum with } p\text{).} \tag{3.9}$$

Thus (P, p) not only represents a configuration of {b} in {s}; it can also be used to convert the representation of a point or frame from {b} coordinates to {s} coordinates.

Now consider a rigid body with two frames attached to it, {d} and {c}. The frame {d} is initially coincident with {s}, and {c} is initially described by (R, r) in {s} (Figure 3.4(a)). Then the body is moved so that {d} moves to {d′}, coincident with a frame {b} described by (P, p) in {s}. Where does {c} end up after this motion? Denoting (R', r') as the configuration of the new frame {c′}, you can verify that

$$R' = PR \tag{3.10}$$

$$r' = Pr + p, \tag{3.11}$$

very similar to Equations (3.8) and (3.9). The difference is that (P, p) is expressed in the same frame as (R, r), so the equations are not viewed as a change of coordinates, but instead as a **rigid-body displacement** (also known as a **rigid-body motion**) that ① rotates {c} according to P and ② translates it by p in {s}. See Figure 3.4(a).

Thus we see that a rotation matrix-vector pair such as (P, p) can be used to do three things:

(i) Represent a configuration of a rigid body in {s} (Figure 3.2).

(ii) Change the reference frame in which a vector or frame is represented (Figure 3.3).

(iii) Displace a vector or a frame (Figure 3.4(a)).

Referring to Figure 3.4(b), note that the rigid-body motion illustrated in Figure 3.4(a), expressed as a rotation followed by a translation, can be obtained by simply rotating the body about a fixed point s by an angle β. This is a planar example of a **screw motion**.[1] The displacement can therefore be parametrized by the three screw coordinates (β, s_x, s_y), where (s_x, s_y) denote the coordinates for the point s (i.e., the screw axis) in the fixed-frame {s}.

[1] If the displacement is a pure translation without rotation, then s lies at infinity.

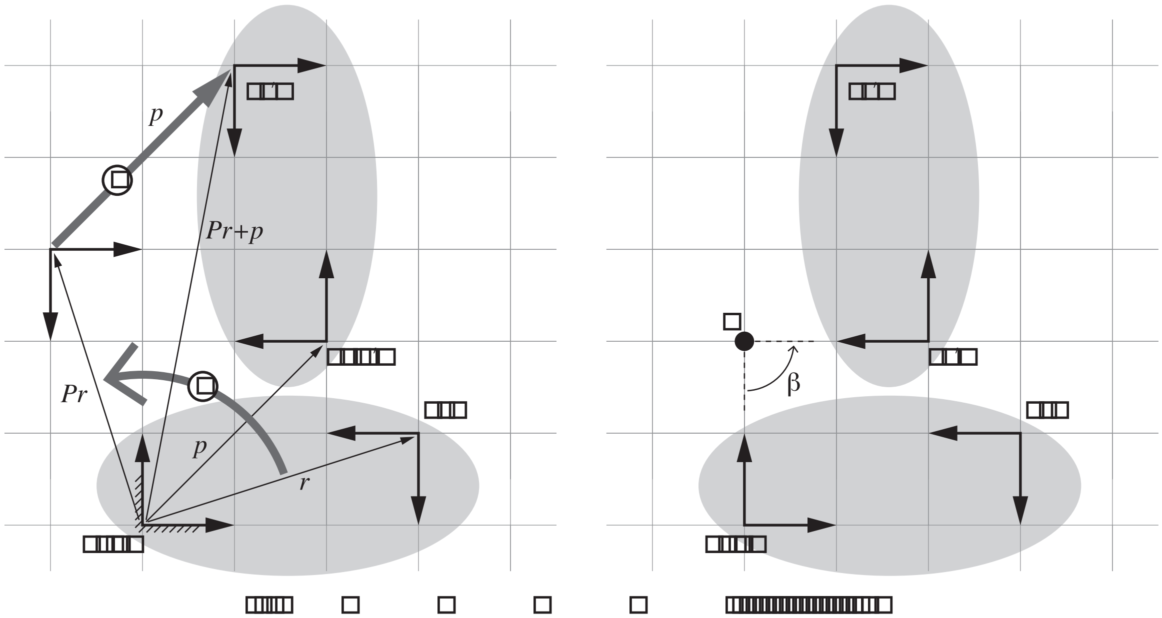


Figure 3.4: (a) The frame {d}, fixed to an elliptical rigid body and initially coincident with {s}, is displaced to {d′} (coincident with the stationary frame {b}), by first rotating according to P then translating according to p, where (P, p) is the representation of {b} in {s}. The same transformation takes the frame {c}, also attached to the rigid body, to {c′}. The transformation marked ① rigidly rotates {c} about the origin of {s}, and then transformation ② translates the frame by p expressed in {s}. (b) Instead of viewing this displacement as a rotation followed by a translation, both rotation and translation can be performed simultaneously. The displacement can be viewed as a rotation of $\beta = 90°$ about a fixed point s.

Another way to represent the screw motion is to consider it the displacement obtained by following a simultaneous angular and linear velocity for a given amount of time. Inspecting Figure 3.4(b), we see that rotating about s with a unit angular velocity ($\omega = 1$ rad/s) means that a point at the origin of the {s} frame moves at two units per second in the $+\hat{x}$ direction of the {s} frame ($v = (v_x, v_y) = (2, 0)$). Packaging these together in a 3-vector as $\mathcal{V} = (\omega, v_x, v_y) = (1, 2, 0)$, we call this the **planar twist** (velocity) corresponding to a unit angular velocity rotation about s. Following this planar twist for time (or angle) $t = \pi/2$ yields the final displacement. We can now express the screw motion (β, s_x, s_y) in the alternate form $\mathcal{V}t = (\pi/2, \pi, 0)$. This form has some advantages, and we call these coordinates the **exponential coordinate** representation of the planar rigid-body displacement.

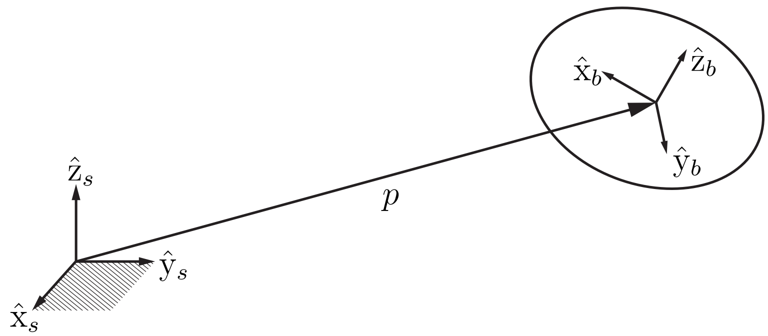


Figure 3.5: Mathematical description of position and orientation.

Remainder of the chapter. In the remainder of this chapter we generalize the concepts above to three-dimensional rigid-body motions. For this purpose consider a rigid body occupying three-dimensional physical space as shown in Figure 3.5. Assume that a length scale for physical space has been chosen, and that both the fixed frame {s} and body frame {b} have been chosen as shown. Throughout this book all reference frames are right-handed, i.e., the unit axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$ always satisfy $\hat{\mathrm{x}} \times \hat{\mathrm{y}} = \hat{\mathrm{z}}$. Denote the unit axes of the fixed frame by $\{\hat{\mathrm{x}}_\mathrm{s}, \hat{\mathrm{y}}_\mathrm{s}, \hat{\mathrm{z}}_\mathrm{s}\}$ and the unit axes of the body frame by $\{\hat{\mathrm{x}}_\mathrm{b}, \hat{\mathrm{y}}_\mathrm{b}, \hat{\mathrm{z}}_\mathrm{b}\}$. Let p denote the vector from the fixed frame origin to the body frame origin. In terms of the fixed frame coordinates, p can be expressed as

$$p = p_1\hat{\mathrm{x}}_\mathrm{s} + p_2\hat{\mathrm{y}}_\mathrm{s} + p_3\hat{\mathrm{z}}_\mathrm{s}. \tag{3.12}$$

The axes of the body frame can also be expressed as

$$\begin{aligned} \hat{\mathrm{x}}_\mathrm{b} &= r_{11}\hat{\mathrm{x}}_\mathrm{s} + r_{21}\hat{\mathrm{y}}_\mathrm{s} + r_{31}\hat{\mathrm{z}}_\mathrm{s} & (3.13)\\ \hat{\mathrm{y}}_\mathrm{b} &= r_{12}\hat{\mathrm{x}}_\mathrm{s} + r_{22}\hat{\mathrm{y}}_\mathrm{s} + r_{32}\hat{\mathrm{z}}_\mathrm{s} & (3.14)\\ \hat{\mathrm{z}}_\mathrm{b} &= r_{13}\hat{\mathrm{x}}_\mathrm{s} + r_{23}\hat{\mathrm{y}}_\mathrm{s} + r_{33}\hat{\mathrm{z}}_\mathrm{s}. & (3.15) \end{aligned}$$

Defining $p \in \mathbb{R}^3$ and $R \in \mathbb{R}^{3\times 3}$ as

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad R = [\hat{\mathrm{x}}_\mathrm{b} \;\; \hat{\mathrm{y}}_\mathrm{b} \;\; \hat{\mathrm{z}}_\mathrm{b}] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \tag{3.16}$$

the twelve parameters given by (R, p) then provide a description of the position and orientation of the rigid body relative to the fixed frame.

Since the orientation of a rigid body has three degrees of freedom, only three of the nine entries in R can be chosen independently. One three-parameter representation of rotations are the exponential coordinates, which define an axis of rotation and the distance rotated about that axis. We leave other popular representations of orientations (three-parameter **Euler angles** and **roll-pitch-yaw angles**, the **Cayley-Rodrigues parameters**, and the **unit quaternions** that use four variables subject to one constraint) to Appendix B.

We then examine six-parameter exponential coordinates for the configuration of a rigid body that arise from integrating a six-dimensional **twist** consisting of the body's angular and linear velocity. This representation follows from the Chasles-Mozzi theorem that states that every rigid-body displacement can be described as a finite rotation and translation about a fixed screw axis.

We conclude with a discussion of forces and moments. Rather than treat these as separate three-dimensional quantities, we merge the moment and force vectors into a six-dimensional **wrench**. The twist and wrench, and rules for manipulating them, form the basis for the kinematic and dynamic analyses in the subsequent chapters.

3.2 Rotations and Angular Velocities

3.2.1 Rotation Matrices

We argued earlier that of the nine entries in the rotation matrix R, only three can be chosen independently. We begin by expressing a set of six explicit constraints on the entries of R. Recall that the three columns of R correspond to the body frame's unit axes $\{\hat{\mathrm{x}}_\mathrm{b}, \hat{\mathrm{y}}_\mathrm{b}, \hat{\mathrm{z}}_\mathrm{b}\}$. The following conditions must therefore be satisfied:

(i) Unit norm condition: $\hat{\mathrm{x}}_\mathrm{b}$, $\hat{\mathrm{y}}_\mathrm{b}$, and $\hat{\mathrm{z}}_\mathrm{b}$ are all of unit norm, or

$$\begin{aligned} r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1 \\ r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1 \\ r_{13}^2 + r_{23}^2 + r_{33}^2 &= 1. \end{aligned} \tag{3.17}$$

(ii) Orthogonality condition: $\hat{\mathrm{x}}_\mathrm{b} \cdot \hat{\mathrm{y}}_\mathrm{b} = \hat{\mathrm{x}}_\mathrm{b} \cdot \hat{\mathrm{z}}_\mathrm{b} = \hat{\mathrm{y}}_\mathrm{b} \cdot \hat{\mathrm{z}}_\mathrm{b} = 0$ (here $\cdot$ denotes the inner product), or

$$\begin{aligned} r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} &= 0 \\ r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} &= 0 \\ r_{11}r_{13} + r_{21}r_{23} + r_{31}r_{33} &= 0. \end{aligned} \tag{3.18}$$

These six constraints can be expressed more compactly as a single set of constraints on the matrix R,

$$R^T R = I, \tag{3.19}$$

where R^T denotes the transpose of R and I denotes the identity matrix.

There is still the matter of accounting for the fact that the frame is right-handed (i.e., $\hat{\mathrm{x}}_\mathrm{b} \times \hat{\mathrm{y}}_\mathrm{b} = \hat{\mathrm{z}}_\mathrm{b}$, where $\times$ denotes the cross-product) rather than left-handed (i.e., $\hat{\mathrm{x}}_\mathrm{b} \times \hat{\mathrm{y}}_\mathrm{b} = -\hat{\mathrm{z}}_\mathrm{b}$); our six equality constraints above do not distinguish between right- and left-handed frames. We recall the following formula for evaluating the determinant of a 3×3 matrix M: denoting the three columns of M by a, b, and c, respectively, its determinant is given by

$$\det M = a^T(b \times c) = c^T(a \times b) = b^T(c \times a). \tag{3.20}$$

Substituting the columns for R into this formula then leads to the constraint

$$\det R = 1. \tag{3.21}$$

Note that if the frame had been left-handed, we would have $\det R = -1$. In summary, the six equality constraints represented by (3.19) imply that $\det R = \pm 1$; imposing the additional constraint $\det R = 1$ means that only right-handed frames are allowed. The constraint $\det R = 1$ does not change the number of independent continuous variables needed to parametrize R.

The set of 3×3 rotation matrices forms the **Special Orthogonal Group** $SO(3)$, which we now formally define:

Definition 3.1. The **Special Orthogonal Group** $SO(3)$, also known as the group of rotation matrices, is the set of all 3×3 real matrices R that satisfy (i) $R^T R = I$ and (ii) $\det R = 1$.

The set of 2×2 rotation matrices is a subgroup of $SO(3)$, and is denoted $SO(2)$.

Definition 3.2. The **Special Orthogonal Group** $SO(2)$ is the set of all 2×2 real matrices R that satisfy (i) $R^T R = I$ and (ii) $\det R = 1$.

From the definition it follows that every $R \in SO(2)$ can be written

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where $\theta \in [0, 2\pi)$. Elements of $SO(2)$ represent planar orientations and elements of $SO(3)$ represent spatial orientations.

3.2.1.1 Properties of Rotation Matrices

The sets of rotation matrices $SO(2)$ and $SO(3)$ are called "groups" because they satisfy the properties required of a mathematical group.[2] Specifically, a group consists of a set of elements and an operation on two elements (matrix multiplication for $SO(n)$) such that, for all A, B in the group, the following properties are satisfied:

- **closure:** AB is also in the group.
- **associativity:** $(AB)C = A(BC)$.
- **identity element existence:** There exists an I in the group (the identity matrix for $SO(n)$) such that $AI = IA = A$.
- **inverse element existence:** There exists an A^{-1} in the group such that $AA^{-1} = A^{-1}A = I$.

[2] More specifically, the $SO(n)$ groups are also called *Lie groups*, where "Lie" is pronounced "lee," because the elements of the group form a differentiable manifold.

Proofs of these properties are given below, using the fact that the identity matrix I is a trivial example of a rotation matrix.

Proposition 3.1. *The inverse of a rotation matrix $R \in SO(3)$ is also a rotation matrix, and it is equal to the transpose of R, i.e., $R^{-1} = R^T$.*

Proof. The condition $R^T R = I$ implies that $R^{-1} = R^T$ and $RR^T = I$. Since $\det R^T = \det R = 1$, R^T is also a rotation matrix. □

Proposition 3.2. *The product of two rotation matrices is a rotation matrix.*

Proof. Given $R_1, R_2 \in SO(3)$, their product $R_1 R_2$ satisfies $(R_1 R_2)^T (R_1 R_2) = R_2^T R_1^T R_1 R_2 = R_2^T R_2 = I$. Further, $\det R_1 R_2 = \det R_1 \cdot \det R_2 = 1$. Thus $R_1 R_2$ satisfies the conditions for a rotation matrix. □

Proposition 3.3. *Multiplication of rotation matrices is associative, $(R_1 R_2) R_3 = R_1 (R_2 R_3)$, but generally not commutative, $R_1 R_2 \neq R_2 R_1$. For the special case of rotation matrices in $SO(2)$, rotations commute.*

Proof. Associativity and non-commutativity follows from properties of matrix multiplication in linear algebra. Commutativity for planar rotations follows from a direct calculation. □

Another important property is that the action of a rotation matrix on a vector (e.g., rotating the vector) does not change the length of the vector.

Proposition 3.4. *For any vector $x \in \mathbb{R}^3$ and $R \in SO(3)$, the vector $y = Rx$ is of the same length as x.*

Proof. This follows from $\|y\|^2 = y^T y = (Rx)^T Rx = x^T R^T Rx = x^T x = \|x\|^2$. □

3.2.1.2 Uses of Rotation Matrices

Analogous to Section 3.1, there are three major uses for a rotation matrix R:

(i) Represent an orientation.

(ii) Change the reference frame in which a vector or a frame is represented.

(iii) Rotate a vector or a frame.

In the first use, R is thought of as representing a frame; in the second and third uses, R is thought of as an operator that acts on a vector or frame (changing its reference frame or rotating it).

To illustrate these uses, refer to Figure 3.6, which shows three different coordinate frames—{a}, {b}, and {c}—representing the same space.[3] These frames have the same origin, since we are only representing orientations, but

[3] In the rest of the book, all coordinate frames will use the same length scale; only their position and orientation may be different.

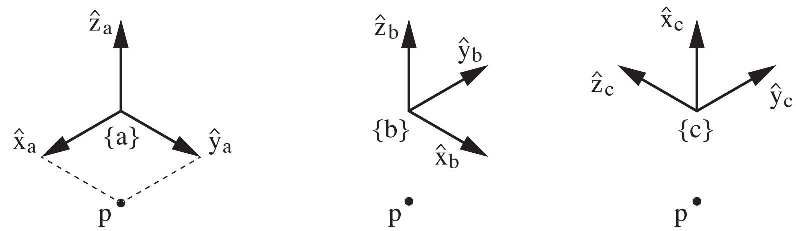


Figure 3.6: The same space and the same point p represented in three different frames with different orientations.

to make the axes clear, the figure shows the same space drawn three times. A point p in the space is also shown. Not shown is a fixed space frame {s}, which is aligned with {a}. The orientations of the three frames relative to {s} can be written

$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix},$$

and the location of the point p in these frames can be written

$$p_a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad p_b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_c = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}.$$

Note that {b} is obtained by rotating {a} about $\hat{z}_a$ by 90°, and {c} is obtained by rotating {b} about $\hat{y}_b$ by −90°. (The direction of positive rotation about an axis, $\theta > 0$, is determined by the direction the fingers of your right hand curl about the axis when you point the thumb of your right hand along the axis.)

Representing an orientation. When we write R_c, we are implicitly referring to the orientation of frame {c} relative to the fixed frame {s}. We can be more explicit about this by writing it as R_{sc}: we are representing the frame of the second subscript, {c}, relative to the frame of the first subscript, {s}. This notation allows us to express a frame relative to a frame that is not {s}; for example, R_{bc} is the orientation of {c} relative to {b}.

If there is no possibility of confusion regarding the frames involved, we may simply write R.

Inspecting Figure 3.6, we see that

$$R_{ac} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_{ca} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

A simple calculation shows that $R_{ac}R_{ca} = I$, i.e., $R_{ac} = R_{ca}^{-1}$, or, equivalently, from Proposition 3.1, $R_{ac} = R_{ca}^T$. In fact, for any two frames {d} and {e},

$$R_{de} = R_{ed}^{-1} = R_{ed}^T.$$

You can verify this fact using any two frames in Figure 3.6.

Changing the reference frame. The rotation matrix R_{ab} represents the orientation of {b} in {a}, and R_{bc} represents the orientation of {c} in {b}. A straightforward calculation shows that the orientation of {c} in {a} can be computed as

$$R_{ac} = R_{ab}R_{bc}. \tag{3.22}$$

In the previous equation, R_{bc} can be viewed as a representation of an orientation, while R_{ab} can be viewed as a mathematical operator that changes the reference frame from {b} to {a}, i.e.,

$$R_{ac} = R_{ab}R_{bc} = \text{change\_reference\_frame\_from\_\{b\}\_to\_\{a\}}\,(R_{bc}).$$

A subscript cancellation rule helps to remember this property. When multiplying two rotation matrices, if the second subscript of the first matrix matches the first subscript of the second matrix, the two subscripts cancel and a change of reference frame is achieved:

$$R_{ab}R_{bc} = R_{a\cancel{b}}R_{\cancel{b}c} = R_{ac}.$$

A rotation matrix is just a collection of three unit vectors, so the reference frame of a vector can also be changed by a rotation matrix using a modified version of the subscript cancellation rule:

$$R_{ab}p_b = R_{a\cancel{b}}p_{\cancel{b}} = p_a.$$

You can verify these properties using the frames and points in Figure 3.6.

Rotating a vector or a frame. The last use of a rotation matrix is to rotate a vector or a frame. Figure 3.7 shows a frame {c} initially aligned with {s} with axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$. If we rotate the {c} frame about a unit axis $\hat{\omega}$ by an amount θ, the new {c′} frame has coordinate axes $\{\hat{\mathrm{x}}', \hat{\mathrm{y}}', \hat{\mathrm{z}}'\}$. The rotation matrix $R = R_{sc'}$ represents the orientation of {c′} relative to {s}, but instead we can think of it as representing the rotation operation itself. Emphasizing our view of R as a rotation operator, instead of as an orientation, we can write

$$R = \mathrm{Rot}(\hat{\omega}, \theta),$$

the operation that rotates the orientation represented by the identity matrix to the orientation represented by R. As we will see in Section 3.2.3.3, for $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$,

$$\mathrm{Rot}(\hat{\omega}, \theta) =$$
$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix},$$

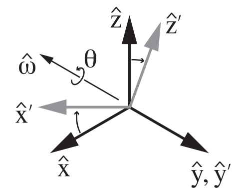


Figure 3.7: A coordinate frame with axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$ is rotated an amount θ about a unit axis $\hat{\omega}$ (which is aligned with $-\hat{\mathrm{y}}$ in this figure). The orientation of the final frame, with axes $\{\hat{\mathrm{x}}', \hat{\mathrm{y}}', \hat{\mathrm{z}}'\}$, is written as R relative to the original frame.

where $\mathrm{s}_\theta = \sin\theta$ and $\mathrm{c}_\theta = \cos\theta$. Any $R \in SO(3)$ can be obtained by rotating from the identity matrix by some θ about some $\hat{\omega}$.

Now say R_{sb} represents some {b} relative to {s}, and we want to rotate {b} by θ about a unit axis $\hat{\omega}$, i.e., $R = \mathrm{Rot}(\hat{\omega}, \theta)$. To be clear about what we mean, we have to specify whether the axis of rotation $\hat{\omega}$ is expressed in {s} coordinates or {b} coordinates. Depending on our choice, the same numerical $\hat{\omega}$ (and therefore the same numerical R) corresponds to different rotation axes in the underlying space, unless the {s} and {b} frames are aligned. Letting {b′} be the new frame after rotating by θ about $\hat{\omega}_s = \hat{\omega}$ (the rotation axis $\hat{\omega}$ is considered to be in the fixed {s} frame), and {b′′} be the new frame after rotating by θ about $\hat{\omega}_b = \hat{\omega}$ (the rotation axis $\hat{\omega}$ is considered to be in the body {b} frame), representations of these new frames can be calculated as

$$R_{sb'} = \text{rotate\_by\_}R\text{\_in\_\{s\}\_frame}\ (R_{sb}) = RR_{sb} \tag{3.23}$$

$$R_{sb''} = \text{rotate\_by\_}R\text{\_in\_\{b\}\_frame}\ (R_{sb}) = R_{sb}R. \tag{3.24}$$

In other words, premultiplying by $R = \mathrm{Rot}(\hat{\omega}, \theta)$ yields a rotation about $\hat{\omega}$ considered to be in the fixed frame, and postmultiplying by R yields a rotation about $\hat{\omega}$ considered to be in the body frame.

Rotating by R in the {s} frame and the {b} frame is illustrated in Figure 3.8.

To rotate a vector v, note that there is only one frame involved, the frame that v is represented in, and therefore $\hat{\omega}$ must be interpreted in this frame. The rotated vector v', in that same frame, is

$$v' = Rv.$$

3.2.1.3 Other Representations of Orientations

Other popular representations of orientations include Euler angles, roll-pitch-yaw angles, the Cayley-Rodrigues parameters, and unit quaternions. These representations, and their relation to rotation matrices, are discussed in Appendix B.

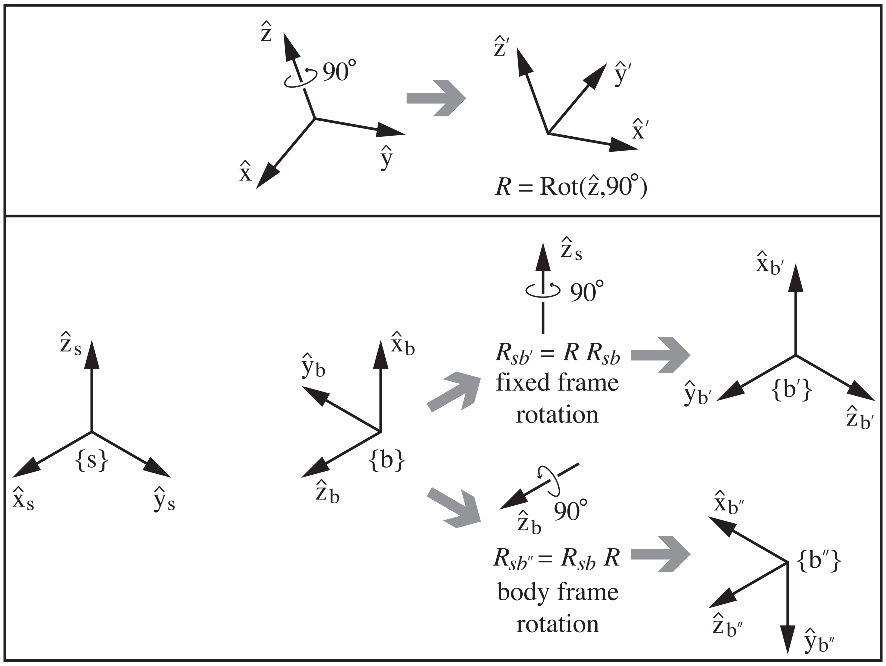


Figure 3.8: (Top) A rotation operator R defined as $R = \mathrm{Rot}(\hat{z}, 90°)$, the orientation of the right frame in the left frame. (Bottom) On the left are shown a fixed frame {s} and a body frame {b}, expressed as R_{sb}. The quantity RR_{sb} rotates {b} to {b′} by rotating by 90° about the fixed frame axis $\hat{\mathrm{z}}_{\mathrm{s}}$. The quantity $R_{sb}R$ rotates {b} to {b″} by rotating by 90° about the body frame axis $\hat{\mathrm{z}}_{\mathrm{b}}$.

3.2.2 Angular Velocity

Referring to Figure 3.9(a), suppose a body frame with unit axes $\{\hat{\mathrm{x}}, \hat{\mathrm{y}}, \hat{\mathrm{z}}\}$ is attached to a rotating body. Let us determine the time derivatives of these unit axes. Beginning with $\dot{\hat{\mathrm{x}}}$, first note that $\hat{\mathrm{x}}$ is of unit length; only the direction of $\hat{\mathrm{x}}$ can vary with time (the same goes for $\hat{\mathrm{y}}$ and $\hat{\mathrm{z}}$). If we examine the body frame at times t and $t+\Delta t$, the change in frame orientation can be described as a rotation of angle $\Delta\theta$ about some unit axis $\hat{\mathrm{w}}$ passing through the origin. The axis $\hat{\mathrm{w}}$ is coordinate-free; it is not yet represented in any particular reference frame.

In the limit as Δt approaches zero, the ratio $\frac{\Delta\theta}{\Delta t}$ becomes the rate of rotation $\dot{\theta}$, and $\hat{\mathrm{w}}$ can similarly be regarded as the instantaneous axis of rotation. In fact, $\hat{\mathrm{w}}$ and $\dot{\theta}$ can be put together to define the **angular velocity** w as follows:

$$\mathrm{w} = \hat{\mathrm{w}}\dot{\theta}. \tag{3.25}$$

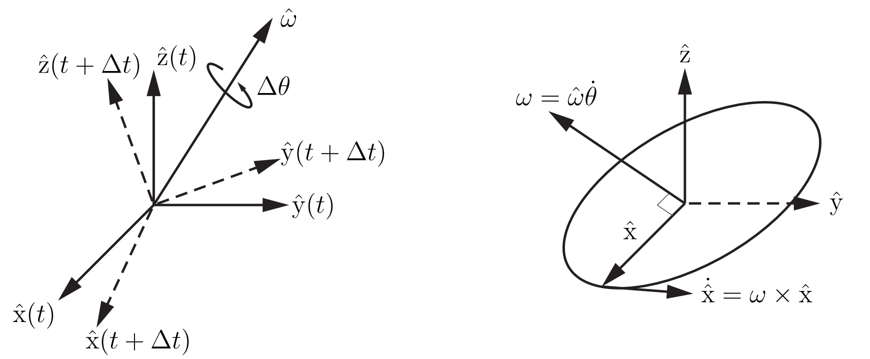


Figure 3.9: (Left) The instantaneous angular velocity vector. (Right) Calculating $\dot{\hat{\mathrm{x}}}$.

Referring to Figure 3.9(b), it should be evident that

$$\begin{aligned} \dot{\hat{\mathrm{x}}} &= \mathrm{w} \times \hat{\mathrm{x}} && (3.26)\\ \dot{\hat{\mathrm{y}}} &= \mathrm{w} \times \hat{\mathrm{y}} && (3.27)\\ \dot{\hat{\mathrm{z}}} &= \mathrm{w} \times \hat{\mathrm{z}}. && (3.28) \end{aligned}$$

To express these equations in coordinates, we have to choose a reference frame in which to represent w. We can choose any reference frame, but two natural choices are the fixed frame $\{\mathrm{s}\}$ and the body frame $\{\mathrm{b}\}$. Let us start with fixed frame $\{\mathrm{s}\}$ coordinates. Let $R(t)$ be the rotation matrix describing the orientation of the body frame with respect to the fixed frame at time t; $\dot{R}(t)$ is its time rate of change. The first column of $R(t)$, denoted $r_1(t)$, describes $\hat{\mathrm{x}}$ in fixed frame coordinates; similarly, $r_2(t)$ and $r_3(t)$ respectively describe $\hat{\mathrm{y}}$ and $\hat{\mathrm{z}}$ in fixed frame coordinates. At a specific time t, let $\omega_s \in \mathbb{R}^3$ be the angular velocity w expressed in fixed frame coordinates. Then Equations (3.26)–(3.28) can be expressed in fixed frame coordinates as

$$\dot{r}_i = \omega_s \times r_i, \quad i = 1, 2, 3.$$

These three equations can be rearranged into the following single 3×3 matrix equation:

$$\dot{R} = [\omega_s \times r_1 \mid \omega_s \times r_2 \mid \omega_s \times r_3] = \omega_s \times R. \qquad (3.29)$$

To eliminate the cross product in Equation (3.29), we introduce some new notation and rewrite $\omega_s \times R$ as $[\omega_s]R$, where $[\omega_s]$ is a 3×3 *skew-symmetric* matrix representation of $\omega_s \in \mathbb{R}^3$:

Definition 3.3. Given a vector $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3$, define

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}. \qquad (3.30)$$

The matrix $[x]$ is a 3×3 **skew-symmetric** matrix representation of x; that is,

$$[x] = -[x]^T.$$

The set of all 3×3 real skew-symmetric matrices is called $so(3)$.[4]

A useful property involving rotations and skew-symmetric matrices is the following:

Proposition 3.5. *Given any $\omega \in \mathbb{R}^3$ and $R \in SO(3)$, the following always holds:*

$$R[\omega]R^T = [R\omega]. \tag{3.31}$$

Proof. Letting r_i^T be the ith row of R,

$$\begin{aligned} R[\omega]R^T &= \begin{bmatrix} r_1^T(\omega \times r_1) & r_1^T(\omega \times r_2) & r_1^T(\omega \times r_3) \\ r_2^T(\omega \times r_1) & r_2^T(\omega \times r_2) & r_2^T(\omega \times r_3) \\ r_3^T(\omega \times r_1) & r_3^T(\omega \times r_2) & r_3^T(\omega \times r_3) \end{bmatrix} \\ &= \begin{bmatrix} 0 & -r_3^T\omega & r_2^T\omega \\ r_3^T\omega & 0 & -r_1^T\omega \\ -r_2^T\omega & r_1^T\omega & 0 \end{bmatrix} \\ &= [R\omega], \end{aligned} \tag{3.32}$$

where the second line makes use of the determinant formula for 3×3 matrices, i.e., if M is a 3×3 matrix with columns $\{a, b, c\}$, then $\det M = a^T(b \times c) = c^T(a \times b) = b^T(c \times a)$. □

With the skew-symmetric notation, we can rewrite Equation (3.29) as

$$[\omega_s]R = \dot{R}. \tag{3.33}$$

We can post-multiply both sides of Equation (3.33) by R^{-1} to get

$$[\omega_s] = \dot{R}R^{-1}. \tag{3.34}$$

Now let ω_b be w expressed in body frame coordinates. To see how to obtain ω_b from ω_s and vice versa, we explicitly write R as R_{sb}. Then ω_s and ω_b are two different vector representations of the same angular velocity w, and by our subscript cancellation rule, $\omega_s = R_{sb}\omega_b$. Therefore

$$\omega_b = R_{sb}^{-1}\omega_s = R^{-1}\omega_s = R^T\omega_s. \tag{3.35}$$

Let us now express this relation in skew-symmetric matrix form:

$$\begin{aligned} [\omega_b] &= [R^T\omega_s] \\ &= R^T[\omega_s]R \quad \text{(by Proposition 3.5)} \\ &= R^T(\dot{R}R^T)R \\ &= R^T\dot{R} = R^{-1}\dot{R}. \end{aligned} \tag{3.36}$$

[4]The set of skew-symmetric matrices $so(3)$ is called the *Lie algebra* of the Lie group $SO(3)$. It consists of all possible $\dot{R}$ when $R = I$.

In summary, we have the following two equations that relate R and $\dot{R}$ to the angular velocity w:

Proposition 3.6. *Let $R(t)$ denote the orientation of the rotating frame as seen from the fixed frame. Denote by* w *the angular velocity of the rotating frame. Then*

$$\begin{aligned} \dot{R}R^{-1} &= [\omega_s] & (3.37) \\ R^{-1}\dot{R} &= [\omega_b], & (3.38) \end{aligned}$$

where $\omega_s \in \mathbb{R}^3$ is the fixed frame vector representation of w *and $[\omega_s] \in so(3)$ is its 3×3 matrix representation, and $\omega_b \in \mathbb{R}^3$ is the body frame vector representation of* w *and $[\omega_b] \in so(3)$ is its 3×3 matrix representation.*

It is important to note that ω_b is *not* the angular velocity relative to a moving frame. Instead, ω_b is the angular velocity relative to the *stationary* frame {b} that is instantaneously coincident with a frame attached to the moving body.

It is also important to note that the fixed-frame angular velocity ω_s *does not depend on the choice of the body frame.* Similarly, the body-frame angular velocity ω_b *does not depend on the choice of the fixed frame.* While Equations (3.37) and (3.38) may appear to depend on both frames (since R and $\dot{R}$ individually depend on both {s} and {b}), the product $\dot{R}R^{-1}$ is independent of {b} and the product $R^{-1}\dot{R}$ is independent of {s}.

Finally, an angular velocity expressed in an arbitrary frame {d} can be represented in another frame {c} if we know the rotation that takes {c} to {d}, using our now-familiar subscript cancellation rule:

$$\omega_c = R_{cd}\omega_d.$$

3.2.3 Exponential Coordinate Representation of Rotation

We now introduce a three-parameter representation for rotations, the **exponential coordinates for rotation**. The exponential coordinates parametrize a rotation matrix in terms of a rotation axis (represented by a unit vector $\hat{\omega}$), together with an angle of rotation θ about that axis; the vector $\hat{\omega}\theta \in \mathbb{R}^3$ then serves as the three-parameter exponential coordinate representation of the rotation. This representation is also called the **axis-angle** representation of a rotation, but we prefer to use the term "exponential coordinates" to emphasize the connection to the upcoming exponential coordinates for rigid-body transformations.

The exponential coordinates for a rotation can be interpreted equivalently as:

- rotating about the axis $\hat{\omega}$ by θ;
- integrating the angular velocity $\hat{\omega}\theta$ for one second; or
- integrating the angular velocity $\hat{\omega}$ for θ seconds.

The latter two views suggest that we consider exponential coordinates in the setting of linear differential equations. Below we briefly review some key results from linear differential equations.

3.2.3.1 Essential Results from Linear Differential Equations

Let us begin with the simple scalar linear differential equation

$$\dot{x}(t) = ax(t), \tag{3.39}$$

where $x(t) \in \mathbb{R}$, $a \in \mathbb{R}$ is constant, and the initial condition $x(0) = x_0$ is assumed given. Equation (3.39) has solution

$$x(t) = e^{at}x_0.$$

It is also useful to remember the series expansion of the exponential function:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \ldots$$

Now consider the vector linear differential equation

$$\dot{x}(t) = Ax(t) \tag{3.40}$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is constant, and the initial condition $x(0) = x_0$ is given. From the earlier scalar result, one can conjecture a solution of the form

$$x(t) = e^{At}x_0 \tag{3.41}$$

where the matrix exponential e^{At} now needs to be defined in a meaningful way. Again mimicking the scalar case, we define the matrix exponential to be

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots \tag{3.42}$$

The first question to be addressed is under what conditions this series converges, so that the matrix exponential is well-defined. It can be shown that if A is constant and finite, this series is always guaranteed to converge to a finite limit; the proof can be found in most texts on ordinary linear differential equations and is not covered here.

The second question is whether Equation (3.41), using Equation (3.42), is indeed a solution to Equation (3.40). Taking the time derivative of $x(t) = e^{At}x_0$,

$$\begin{aligned}
\dot{x}(t) &= \left(\frac{d}{dt}e^{At}\right)x_0 \\
&= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots\right)x_0 \\
&= \left(A + A^2t + \frac{A^3t^2}{2!} + \ldots\right)x_0 \qquad (3.43)\\
&= Ae^{At}x_0 \\
&= Ax(t),
\end{aligned}$$

which proves that $x(t) = e^{At}x_0$ is indeed a solution. That this is a unique solution follows from the basic existence and uniqueness result for linear ordinary differential equations, which we invoke here without proof.

While $AB \neq BA$ for arbitrary square matrices A and B, it is always true that

$$Ae^{At} = e^{At}A \tag{3.44}$$

for any square A and scalar t. You can verify this directly using the series expansion for the matrix exponential. Therefore, in line four of Equation (3.43), A could also have been factored to the right, i.e.,

$$\dot{x}(t) = e^{At}Ax_0.$$

While the matrix exponential e^{At} is defined as an infinite series, closed-form expressions are often available. For example, if A can be expressed as $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned} e^{At} &= I + At + \frac{(At)^2}{2!} + \ldots \\ &= I + (PDP^{-1})t + (PDP^{-1})(PDP^{-1})\frac{t^2}{2!} + \ldots \\ &= P(I + Dt + \frac{(Dt)^2}{2!} + \ldots)P^{-1} \\ &= Pe^{Dt}P^{-1}. \end{aligned} \tag{3.45}$$

If moreover D is diagonal, i.e., $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$, then its matrix exponential is particularly simple to evaluate:

$$e^{Dt} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix}. \tag{3.46}$$

We summarize the results above in the following proposition.

Proposition 3.7. *The linear differential equation $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0$, where $A \in \mathbb{R}^{n\times n}$ is constant and $x(t) \in \mathbb{R}^n$, has solution*

$$x(t) = e^{At}x_0 \tag{3.47}$$

where

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \ldots. \tag{3.48}$$

The matrix exponential e^{At} further satisifies the following properties:

(i) $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.

(ii) *If $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$, then $e^{At} = Pe^{Dt}P^{-1}$.*

(iii) *If* $AB = BA$, *then* $e^A e^B = e^{A+B}$.

(iv) $(e^A)^{-1} = e^{-A}$.

The third property can be established by expanding the exponentials and comparing terms. The fourth property follows by setting $B = -A$ in the third property.

A final linear algebraic result useful in finding closed-form expressions for e^{At} is the Cayley-Hamilton Theorem, which we state here without proof:

Proposition 3.8. *Let* $A \in \mathbb{R}^{n \times n}$ *be constant, with characteristic polynomial*

$$p(s) = \det(sI - A) = s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0,$$

and define $p(A)$ *as*

$$p(A) = A^n + c_{n-1}A^{n-1} + \ldots + c_1 A + c_0 I.$$

Then $p(A) = 0$.

3.2.3.2 Exponential Coordinates of Rotations

The exponential coordinates of a rotation can be viewed equivalently as (1) a unit axis of rotation $\hat{\omega}$ ($\hat{\omega} \in \mathbb{R}^3, \|\hat{\omega}\| = 1$) together with a rotation angle about the axis $\theta \in \mathbb{R}$, or (2) as the three-vector obtained by multiplying the two together, $\hat{\omega}\theta \in \mathbb{R}^3$. When we represent the motion of a robot joint in the next chapter, the first view has the advantage of separating the description of the joint axis from the motion θ about the axis.

Referring to Figure 3.10, suppose a three-dimensional vector $p(0)$ is rotated by θ about $\hat{\omega}$ to $p(\theta)$; here we assume all quantities are expressed in fixed frame coordinates. This rotation can be achieved by imagining that $p(0)$ rotates at a constant rate of 1 rad/s (since $\hat{\omega}$ is unit) from time $t = 0$ to $t = \theta$. Let $p(t)$ denote this path. The velocity of $p(t)$, denoted $\dot{p}$, is then given by

$$\dot{p} = \hat{\omega} \times p. \tag{3.49}$$

To see why this is true, let ϕ be the angle between $p(t)$ and $\hat{\omega}$. Observe that p traces a circle of radius $\|p\| \sin\phi$ about the $\hat{\omega}$-axis. Then $\dot{p} = \hat{\omega} \times p$ is tangent to the path with magnitude $\|p\| \sin\phi$, which is exactly Equation (3.49).

The differential equation (3.49) can be expressed as

$$\dot{p} = [\hat{\omega}]p \tag{3.50}$$

with initial condition $p(0)$. This is a linear differential equation of the form $\dot{x} = Ax$ that we studied earlier; its solution is given by

$$p(t) = e^{[\hat{\omega}]t}p(0).$$

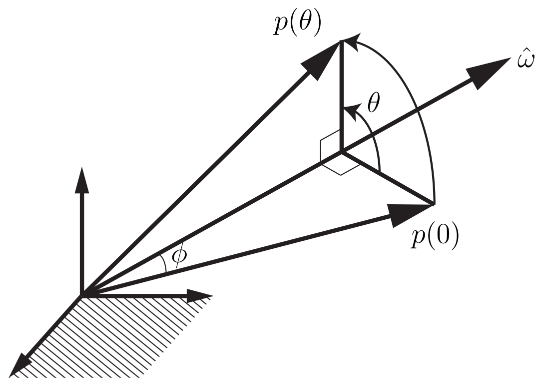


Figure 3.10: The vector $p(0)$ is rotated by an angle θ about the axis $\hat\omega$, to $p(\theta)$.

Since t and θ are interchangeable, the equation above can also be written

$$p(\theta) = e^{[\hat\omega]\theta} p(0).$$

We now derive a closed-form expression for $e^{[\hat\omega]\theta}$. Here we make use of the Cayley-Hamilton Theorem. First, the characteristic polynomial associated with the 3×3 matrix $[\hat\omega]$ is given by

$$p(s) = \det(sI - [\hat\omega]) = s^3 + s.$$

The Cayley-Hamilton Theorem then implies $[\hat\omega]^3 + [\hat\omega] = 0$, or

$$[\hat\omega]^3 = -[\hat\omega].$$

Let us now expand the matrix exponential $e^{[\hat\omega]\theta}$ in series form. Replacing $[\hat\omega]^3$ by $-[\hat\omega]$, $[\hat\omega]^4$ by $-[\hat\omega]^2$, $[\hat\omega]^5$ by $-[\hat\omega]^3 = [\hat\omega]$, and so on, we obtain

$$\begin{aligned} e^{[\hat\omega]\theta} &= I + [\hat\omega]\theta + [\hat\omega]^2\frac{\theta^2}{2!} + [\hat\omega]^3\frac{\theta^3}{3!} + \ldots \\ &= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat\omega] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat\omega]^2. \end{aligned}$$

Now recall the series expansions for $\sin\theta$ and $\cos\theta$:

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots \end{aligned}$$

The exponential $e^{[\hat\omega]\theta}$ therefore simplifies to the following:

Proposition 3.9. *Given a vector $\hat\omega\theta \in \mathbb{R}^3$, such that θ is any scalar and $\hat\omega \in \mathbb{R}^3$ is a unit vector,*

$$\mathrm{Rot}(\hat\omega, \theta) = e^{[\hat\omega]\theta} = I + \sin\theta\,[\hat\omega] + (1 - \cos\theta)[\hat\omega]^2 \in SO(3). \tag{3.51}$$

This formula provides the matrix exponential of $[\hat\omega]\theta \in so(3)$.

This formula is also known as **Rodrigues' formula** for rotations.

We have shown how to use the matrix exponential to construct a rotation matrix from a rotation axis $\hat{\omega}$ and an angle θ. Further, the quantity $e^{[\hat{\omega}]\theta}p$ amounts to rotating $p \in \mathbb{R}^3$ about the fixed-frame axis $\hat{\omega}$ by an angle θ. Similarly, considering that a rotation matrix R consists of three column vectors, the rotation matrix $R' = e^{[\hat{\omega}]\theta}R = \mathrm{Rot}(\hat{\omega},\theta)R$ is the orientation achieved by rotating R by θ about the axis $\hat{\omega}$ in the fixed frame. Reversing the order of matrix multiplication, $R'' = Re^{[\hat{\omega}]\theta} = R\,\mathrm{Rot}(\hat{\omega},\theta)$ is the orientation achieved by rotating R by θ about $\hat{\omega}$ in the body frame.

Example 3.1. The frame {b} in Figure 3.11 is obtained by rotating from an initial orientation aligned with the fixed frame {s} about a unit axis $\hat{\omega} = (0, 0.866, 0.5)^T$ by an angle of $\theta = 30° = 0.524$ rad. The rotation matrix representation of {b} can be calculated as

$$\begin{aligned}
R &= e^{[\hat{\omega}]\theta} \\
&= I + \sin\theta[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \\
&= I + 0.5\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix} + 0.134\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix}^2 \\
&= \begin{bmatrix} 0.866 & -0.250 & 0.433 \\ 0.250 & 0.967 & 0.058 \\ -0.433 & 0.058 & 0.900 \end{bmatrix}.
\end{aligned}$$

The frame {b} can be represented by R or by its exponential coordinates $\hat{\omega} = (0, 0.866, 0.5)^T$ and $\theta = 0.524$ rad, i.e., $\hat{\omega}\theta = (0, 0.453, 0.262)^T$.

If {b} is then rotated by $-\theta = -0.524$ rad about the same fixed-frame axis $\hat{\omega}$, i.e.,

$$R' = e^{-[\hat{\omega}]\theta}R,$$

then we would find $R' = I$, as expected; the frame has rotated back to the identity (aligned with the {s} frame). On the other hand, if {b} were to be rotated by $-\theta$ about $\hat{\omega}$ in the body frame (this axis is different from $\hat{\omega}$ in the fixed frame), the new orientation would not be aligned with {s}:

$$R'' = Re^{-[\hat{\omega}]\theta} \neq I.$$

Our next task is to show that for any rotation matrix $R \in SO(3)$, one can always find a unit vector $\hat{\omega}$ and scalar θ such that $R = e^{[\hat{\omega}]\theta}$.

3.2.3.3 Matrix Logarithm of Rotations

If $\hat{\omega}\theta \in \mathbb{R}^3$ represents the exponential coordinates of a rotation matrix R, then the skew-symmetric matrix $[\hat{\omega}\theta] = [\hat{\omega}]\theta$ is the **matrix logarithm** of the rotation R. The matrix logarithm is the inverse of the matrix exponential. Just as the

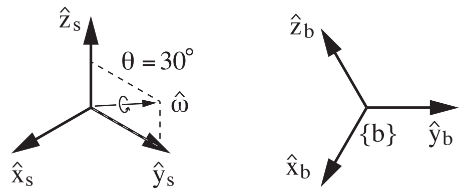


Figure 3.11: The frame {b} is obtained by rotating from {s} by $\theta = 30°$ about $\hat{\omega} = (0, 0.866, 0.5)^T$.

matrix exponential "integrates" the matrix representation of an angular velocity $[\hat{\omega}]\theta \in so(3)$ for one second to give an orientation $R \in SO(3)$, the matrix logarithm "differentiates" an $R \in SO(3)$ to find the matrix representation of a constant angular velocity $[\hat{\omega}]\theta \in so(3)$ which, if integrated for one second, rotates a frame from I to R. In other words,

$$\begin{array}{rlcl} \exp: & [\hat{\omega}]\theta \in so(3) & \rightarrow & R \in SO(3) \\ \log: & R \in SO(3) & \rightarrow & [\hat{\omega}]\theta \in so(3) \end{array}$$

To derive the matrix logarithm, let us expand each of the entries for $e^{[\hat{\omega}]\theta}$ in Equation (3.51),

$$\left[\begin{array}{ccc} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{array}\right], \quad (3.52)$$

where $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)^T$, and we use the shorthand notation $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. Setting the above equal to the given $R \in SO(3)$ and subtracting the transpose from both sides leads to the following:

$$\begin{array}{rcl} r_{32} - r_{23} & = & 2\hat{\omega}_1 \sin\theta \\ r_{13} - r_{31} & = & 2\hat{\omega}_2 \sin\theta \\ r_{21} - r_{12} & = & 2\hat{\omega}_3 \sin\theta. \end{array}$$

Therefore, as long as $\sin\theta \neq 0$ (or equivalently, θ is not an integer multiple of π), we can write

$$\begin{array}{rcl} \hat{\omega}_1 & = & \dfrac{1}{2\sin\theta}(r_{32} - r_{23}) \\ \hat{\omega}_2 & = & \dfrac{1}{2\sin\theta}(r_{13} - r_{31}) \\ \hat{\omega}_3 & = & \dfrac{1}{2\sin\theta}(r_{21} - r_{12}). \end{array}$$

The above equations can also be expressed in skew-symmetric matrix form as

$$[\hat{\omega}] = \left[\begin{array}{ccc} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{array}\right] = \frac{1}{2\sin\theta}\left(R - R^T\right). \quad (3.53)$$

Recall that $\hat{\omega}$ represents the axis of rotation for the given R. Because of the $\sin\theta$ term in the denominator, $[\hat{\omega}]$ is not well defined if θ is an integer multiple of π.[5] We address this situation next, but for now let us assume this is not the case and find an expression for θ. Setting R equal to (3.52) and taking the trace of both sides (recall that the trace of a matrix is the sum of its diagonal entries),

$$\mathrm{tr}\, R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta. \tag{3.54}$$

The above follows since $\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$. For any θ satisfying $1 + 2\cos\theta = \mathrm{tr}\, R$ such that θ is not an integer multiple of π, R can be expressed as the exponential $e^{[\hat{\omega}]\theta}$ with $[\hat{\omega}]$ as given in Equation (3.53).

Let us now return to the case $\theta = k\pi$, where k is some integer. When k is an even integer, regardless of $\hat{\omega}$, we have rotated back to $R = I$, so the vector $\hat{\omega}$ is undefined. When k is an odd integer (corresponding to $\theta = \pm\pi, \pm 3\pi, \ldots$, which in turn implies tr $R = -1$), the exponential formula (3.51) simplifies to

$$R = e^{[\hat{\omega}]\pi} = I + 2[\hat{\omega}]^2. \tag{3.55}$$

The three diagonal terms of Equation (3.55) can be manipulated to

$$\hat{\omega}_i = \pm\sqrt{\frac{r_{ii} + 1}{2}}, \quad i = 1, 2, 3. \tag{3.56}$$

The off-diagonal terms lead to the following three equations:

$$\begin{aligned} 2\hat{\omega}_1\hat{\omega}_2 &= r_{12} \\ 2\hat{\omega}_2\hat{\omega}_3 &= r_{23} \\ 2\hat{\omega}_1\hat{\omega}_3 &= r_{13}, \end{aligned} \tag{3.57}$$

From Equation (3.55) we also know that R must be symmetric: $r_{12} = r_{21}$, $r_{23} = r_{32}$, $r_{13} = r_{31}$. Both Equations (3.56) and (3.57) may be necessary to obtain a solution for $\hat{\omega}$. Once a solution $\hat{\omega}$ has been found, then $R = e^{[\hat{\omega}]\theta}$, where $\theta = \pm\pi, \pm 3\pi, \ldots$.

From the above it can be seen that solutions for θ exist at 2π intervals. If we restrict θ to the interval $[0, \pi]$, then the following algorithm can be used to compute the matrix logarithm of the rotation matrix $R \in SO(3)$.

Algorithm: Given $R \in SO(3)$, find a $\theta \in [0, \pi]$ and a unit rotation axis $\hat{\omega} \in \mathbb{R}^3, \|\hat{\omega} = 1\|$, such that $e^{[\hat{\omega}]\theta} = R$. The vector $\hat{\omega}\theta \in \mathbb{R}^3$ comprises the exponential coordinates for R and the skew-symmetric matrix $[\hat{\omega}]\theta \in so(3)$ is a matrix logarithm of R.

(i) If $R = I$, then $\theta = 0$ and $\hat{\omega}$ is undefined.

[5] A singularity such as this is unavoidable for any three-parameter representation of rotation. Euler angles and roll-pitch-yaw angles suffer similar singularities.

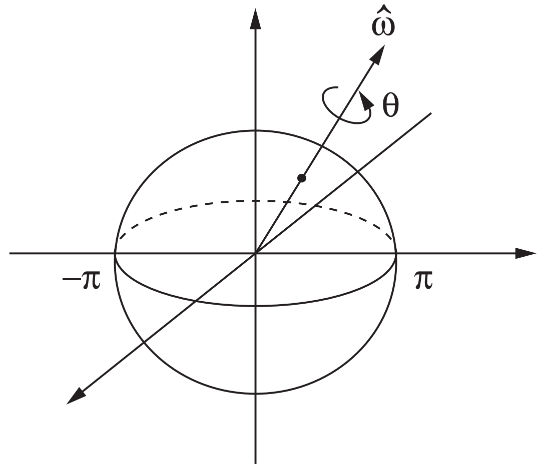


Figure 3.12: *SO(3)* as a solid ball of radius π.

(ii) If $\operatorname{tr} R = -1$, then $\theta = \pi$. Set $\hat{\omega}$ to any of the following three vectors that is a feasible solution:

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix} \tag{3.58}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1+r_{22} \\ r_{32} \end{bmatrix} \tag{3.59}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{11})}} \begin{bmatrix} 1+r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}. \tag{3.60}$$

(iii) Otherwise $\theta = \cos^{-1}\left(\frac{\operatorname{tr} R-1}{2}\right) \in [0,\pi)$ and

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^T). \tag{3.61}$$

Since every $R \in SO(3)$ satisfies one of the three cases in the algorithm, for every R there exists a set of exponential coordinates $\hat{\omega}\theta$.

The formula for the logarithm suggests a picture of the rotation group $SO(3)$ as a solid ball of radius π (see Figure 3.12): given a point $r \in \mathbb{R}^3$ in this solid ball, let $\hat{\omega} = r/\|r\|$ be the unit axis in the direction from the origin to r and and $\theta = \|r\|$ be the distance from the origin to r, so that $r = \hat{\omega}\theta$. The rotation matrix corresponding to r can then be regarded as a rotation about the axis $\hat{\omega}$ by an angle θ. For any $R \in SO(3)$ such that $\operatorname{tr} R \neq -1$, there exists a unique r in the interior of the solid ball such that $e^{[r]} = R$. In the event that $\operatorname{tr} R = -1$, $\log R$ is given by two antipodal points on the surface of this solid ball. That is, if there exists some r such that $R = e^{[r]}$ with $\|r\| = \pi$, then $R = e^{[-r]}$ also holds; both r and $-r$ correspond to the same rotation R.

3.3 Rigid-Body Motions and Twists

In this section we derive representations for rigid-body configurations and velocities that extend, but otherwise are analogous to, those in Section 3.2 for rotations and angular velocities. In particular, the homogeneous transformation matrix T is analogous to the rotation matrix R; a screw axis $\mathcal{S}$ is analogous to a rotation axis $\hat{\omega}$; a twist $\mathcal{V}$ can be expressed as $\mathcal{S}\dot{\theta}$ and is analogous to an angular velocity $\omega = \hat{\omega}\dot{\theta}$; and exponential coordinates $\mathcal{S}\theta \in \mathbb{R}^6$ for rigid-body motions are analogous to exponential coordinates $\hat{\omega}\theta \in \mathbb{R}^3$ for rotations.

3.3.1 Homogeneous Transformation Matrices

We now consider representations for the combined orientation and position of a rigid body. A natural choice would be to use a rotation matrix $R \in SO(3)$ to represent the orientation of {b} in {s} and a vector $p \in \mathbb{R}^3$ to represent the origin of {b} in {s}. Rather than identifying R and p separately, we package them into a single matrix as follows.

Definition 3.4. The **Special Euclidean Group** $SE(3)$, also known as the group of **rigid-body motions** or **homogeneous transformations** in $\mathbb{R}^3$, is the set of all 4×4 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.62}$$

where $R \in SO(3)$ and $p \in \mathbb{R}^3$ is a column vector.

An element $T \in SE(3)$ will sometimes be denoted (R, p). We begin this section by establishing some basic properties of $SE(3)$, and explaining why we package R and p into this matrix form.

Many of the robotic mechanisms we have encountered thus far are planar. With planar rigid-body motions in mind, we make the following definition:

Definition 3.5. The **Special Euclidean Group** $SE(2)$ is the set of all 3×3 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \tag{3.63}$$

where $R \in SO(2)$, $p \in \mathbb{R}^2$, and 0 denotes a row vector of two zeros.

A matrix $T \in SE(2)$ is always of the form

$$T = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\theta \in [0, 2\pi)$.

3.3.1.1 Properties of Transformation Matrices

We now list some basic properties of transformation matrices, which can be proven by calculation. First, the identity I is a trivial example of a transformation matrix. The first three properties confirm that $SE(3)$ is a group.

Proposition 3.10. *The inverse of a transformation matrix $T \in SE(3)$ is also a transformation matrix, and has the following form*

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}. \tag{3.64}$$

Proposition 3.11. *The product of two transformation matrices is also a transformation matrix.*

Proposition 3.12. *Multiplication of transformation matrices is associative, $(T_1T_2)T_3 = T_1(T_2T_3)$, but generally not commutative, $T_1T_2 \neq T_2T_1$.*

Before stating the next proposition, we note that just as in Section 3.1, it is often useful to calculate the quantity $Rx + p$, where $x \in \mathbb{R}^3$ and (R, p) represents T. If we append a '1' to x, making it a four-dimensional vector, this computation can be performed as a single matrix multiplication:

$$T \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}. \tag{3.65}$$

The vector $(x^T, 1)^T$ is the representation of x in **homogeneous coordinates**, and accordingly $T \in SE(3)$ is called a homogenous transformation. When, by an abuse of notation, we write Tx, we mean $Rx + p$.

Proposition 3.13. *Given $T = (R, p) \in SE(3)$ and $x, y \in \mathbb{R}^3$, the following hold:*

(i) $\|Tx - Ty\| = \|x - y\|$, *where $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^3$, i.e., $\|x\| = \sqrt{x^T x}$.*

(ii) $\langle Tx - Tz, Ty - Tz\rangle = \langle x - z, y - z\rangle$ *for all $z \in \mathbb{R}^3$, where $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product in $\mathbb{R}^3$, i.e., $\langle x, y\rangle = x^T y$.*

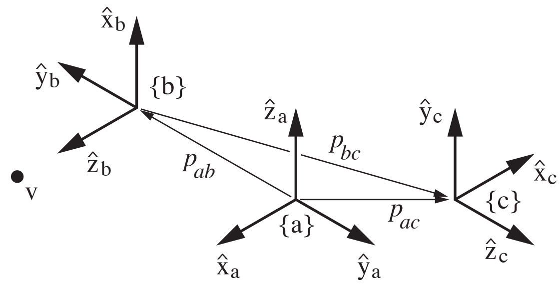


Figure 3.13: Three reference frames in space, and a point v that can be represented in {b} as $v_b = (0, 0, 1.5)^T$.

In Proposition 3.13, T is regarded as a transformation on points in $\mathbb{R}^3$, i.e., T transforms a point x to Tx. The first property then asserts that T preserves distances, while the second asserts that T preserves angles. Specifically, if $x, y, z \in \mathbb{R}^3$ represent the three vertices of a triangle, then the triangle formed by the transformed vertices $\{Tx, Ty, Tz\}$ has the same set of lengths and angles as those of the triangle $\{x, y, z\}$ (the two triangles are said to be *isometric*). One can easily imagine taking $\{x, y, z\}$ to be the points on a rigid body, in which case $\{Tx, Ty, Tz\}$ represents a displaced version of the rigid body. It is in this sense that $SE(3)$ can be identified with the rigid-body motions.

3.3.1.2 Uses of Transformation Matrices

As with rotation matrices, there are three major uses for a transformation matrix T:

(i) Represent the configuration (position and orientation) of a rigid body.

(ii) Change the reference frame in which a vector or frame is represented.

(iii) Displace a vector or frame.

In the first use, T is thought of as representing the configuration of a frame; in the second and third uses, T is thought of as an operator that acts to change the reference frame or to move a vector or a frame.

To illustrate these uses, we refer to the three reference frames {a}, {b}, and {c}, and the point v, in Figure 3.13. These frames are chosen so that the alignment of their axes is clear, allowing visual confirmation of calculations.

Representing a configuration. The fixed frame {s} is coincident with {a}, and the frames {a}, {b}, and {c}, represented by $T_{sa} = (R_{sa}, p_{sa})$, $T_{sb} = (R_{sb}, p_{sb})$, $T_{sc} = (R_{sc}, p_{sc})$, respectively, can be expressed relative to {s} by the rotations

$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and the location of the origin of each frame relative to {s} can be written

$$p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \quad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Since {a} is collocated with {s}, the transformation matrix T_{sa} constructed from (R_{sa}, p_{sa}) is the identity matrix.

Any frame can be expressed relative to any other frame, not just {s}; for example, $T_{bc} = (R_{bc}, p_{bc})$ represents {b} relative to {c}:

$$R_{bc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \quad p_{bc} = \begin{bmatrix} 0 \\ -3 \\ -1 \end{bmatrix}.$$

It can also be shown, using Proposition 3.10, that

$$T_{de} = T_{ed}^{-1}$$

for any two frames {d} and {e}.

Changing the reference frame of a vector or a frame. By a subscript cancellation rule analogous to that for rotations, for any three reference frames {a}, {b}, and {c}, and any vector v expressed in {b} as v_b,

$$\begin{aligned} T_{ab}T_{bc} &= T_{a\cancel{b}}T_{\cancel{b}c} = T_{ac} \\ T_{ab}v_b &= T_{a\cancel{b}}v_{\cancel{b}} = v_a, \end{aligned}$$

where v_a is the vector v expressed in {a}.

Displacing (rotating and translating) a vector or a frame. A transformation matrix T, viewed as the pair $(R, p) = (\mathrm{Rot}(\hat{\omega}, \theta), p)$, can act on a frame T_{sb} by rotating it by θ about an axis $\hat{\omega}$ and translating it by p. Whether we pre-multiply or post-multiply T_{sb} by the operator T determines whether the $\hat{\omega}$ axis and p are interpreted in the fixed frame {s} or the body frame {b}:

$$T_{sb'} = TT_{sb} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & Rp_{sb} + p \\ 0 & 1 \end{bmatrix} \quad \text{(fixed frame)} \tag{3.66}$$

$$T_{sb''} = T_{sb}T = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \\ 0 & 1 \end{bmatrix} \quad \text{(body frame)}. \tag{3.67}$$

The fixed-frame transformation (pre-multiplication by T) can be interpreted as first rotating the {b} frame by θ about an axis $\hat{\omega}$ in the {s} frame (this rotation will cause the origin of {b} to move if it is not coincident with the origin of {s}), then translating it by p in the {s} frame to get the {b′} frame. The

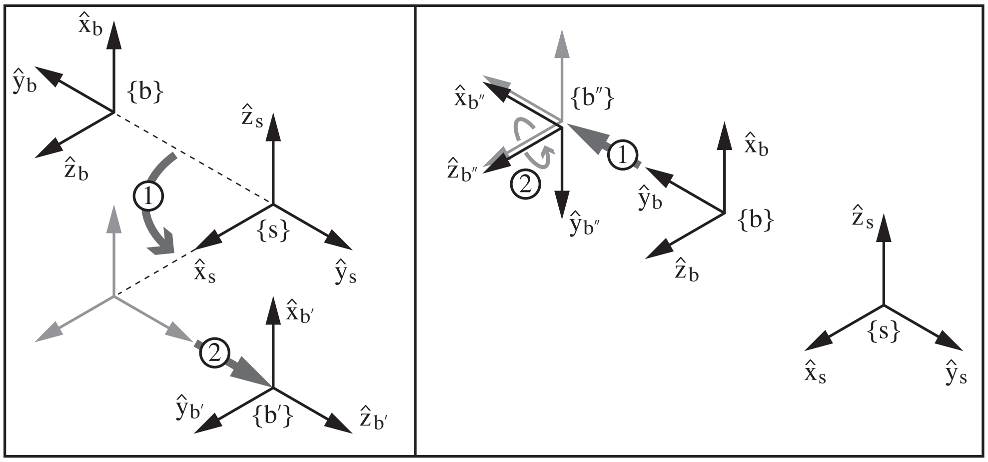


Figure 3.14: Fixed-frame and body-frame transformations corresponding to $\hat{\omega} = (0,0,1)^T$, $\theta = 90°$, and $p = (0,2,0)^T$. (Left) The frame {b} rotated by 90° about $\hat{z}_s$ and then translated by two units in $\hat{y}_s$, resulting in the new frame {b′}. (Right) The frame {b} translated by two units in $\hat{y}_b$ and then rotated by 90° about the $\hat{z}$ axis of the body frame, resulting in the new frame {b″}.

body-frame transformation (post-multiplication by T) can be interpreted as first translating {b} by p considered to be in the {b} frame, then rotating about $\hat{\omega}$ in this new body frame (this does not move the origin of the frame) to get {b″}. Fixed-frame and body-frame transformations are illustrated in Figure 3.14 for a transformation T with $\hat{\omega} = (0,0,1)^T$, $\theta = 90°$, and $p = (0,2,0)^T$, yielding

$$T = (\mathrm{Rot}(\hat{\omega},\theta), p) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Beginning with the frame {b} represented by

$$T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the new frame {b′} achieved by a fixed-frame transformation TT_{sb} and the new frame {b″} achieved by a body-frame transformation $T_{sb}T$ are

$$TT_{sb} = T_{sb'} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{sb}T = T_{sb''} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

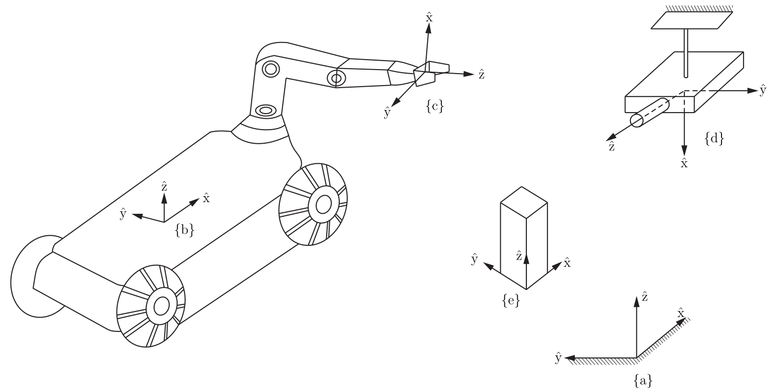


Figure 3.15: Assignment of reference frames.

Example 3.2. Figure 3.15 shows a robot arm mounted on a wheeled mobile platform, and a camera fixed to the ceiling. Frames {b} and {c} are respectively attached to the wheeled platform and the end-effector of the robot arm, and frame {d} is attached to the camera. A fixed frame {a} has been established, and the robot must pick up the object with body frame {e}. Suppose that the transformations T_{db} and T_{de} can be calculated from measurements obtained with the camera. The transformation T_{bc} can be calculated using the arm's joint angle measurements. The transformation T_{ad} is assumed to be known in advance. Suppose these known transformations are given as follows:

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{de} = \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{ad} = \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{bc} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 30 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In order to calculate how to move the robot arm to pick up the object, the configuration of the object relative to the robot hand, T_{ce}, must be determined.

We know that

$$T_{ab}T_{bc}T_{ce} = T_{ad}T_{de},$$

where the only quantity besides T_{ce} not given to us directly is T_{ab}. However, since $T_{ab} = T_{ad}T_{db}$, we can determine T_{ce} as follows:

$$T_{ce} = (T_{ad}T_{db}T_{bc})^{-1}\, T_{ad}T_{de}.$$

From the given transformations,

$$\begin{aligned}
T_{ad}T_{de} &= \begin{bmatrix} 1 & 0 & 0 & 280 \\ 0 & 1 & 0 & -50 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
T_{ad}T_{db}T_{bc} &= \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 230 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 160 \\ 1 & 0 & 0 & 75 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
(T_{ad}T_{db}T_{bc})^{-1} &= \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 70/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 390/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

from which T_{ce} is evaluated to be

$$T_{ce} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -260/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 130/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3.3.2 Twists

We now consider both the linear and angular velocity of a moving frame. As before, denote by {s} and {b} the fixed (space) and moving (body) frames, respectively, and let

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix} \tag{3.68}$$

denote the homogeneous transformation of {b} as seen from {s} (to keep the notation uncluttered, for the time being we write T instead of the usual T_{sb}).

In Section 3.2.2 we discovered that pre- or post-multiplying $\dot{R}$ by R^{-1} results in a skew-symmetric representation of the angular velocity vector, either in fixed or body frame coordinates. One might reasonably ask if a similar property carries over to $\dot{T}$, i.e., whether $T^{-1}\dot{T}$ and $\dot{T}T^{-1}$ carry similar physical interpretations.

Let us first see what happens when we pre-multiply $\dot{T}$ by T^{-1}:

$$\begin{aligned} T^{-1}\dot{T} &= \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R^T \dot{R} & R^T \dot{p} \\ 0 & 0 \end{bmatrix} \qquad (3.69) \\ &= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}. \qquad (3.70) \end{aligned}$$

Recall that $R^T\dot{R} = [\omega_b]$ is just the skew-symmetric matrix representation of the angular velocity expressed in {b} coordinates. Also, $\dot{p}$ is the linear velocity of the origin of {b} expressed in the fixed frame {s}, and $R^T\dot{p} = v_b$ is this linear velocity expressed in the frame {b}. Putting these two observations together, we can conclude that $T^{-1}\dot{T}$ represents the linear and angular velocity of the moving frame relative to the stationary frame {b} currently aligned with the moving frame.

The previous calculation of $T^{-1}\dot{T}$ suggests that it is reasonable to merge ω_b and v_b into a single six-dimensional velocity vector. We define the **spatial velocity in the body frame**, or simply the **body twist**[6], to be

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} \in \mathbb{R}^6. \qquad (3.71)$$

Just as it is convenient to have a skew-symmetric matrix representation of an angular velocity vector, it is convenient to have a matrix representation of a twist, as shown in Equation (3.70). We overload the $[\cdot]$ notation, writing

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3), \qquad (3.72)$$

where $[\omega_b] \in so(3)$ and $v_b \in \mathbb{R}^3$. The set of all 4×4 matrices of this form is called $se(3)$, the matrix representation of velocities associated with the rigid-body configurations $SE(3)$.[7]

Now that we have a physical interpretation for $T^{-1}\dot{T}$, let us evaluate $\dot{T}T^{-1}$:

$$\begin{aligned} \dot{T}T^{-1} &= \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \dot{R}R^T & \dot{p} - \dot{R}R^T p \\ 0 & 0 \end{bmatrix} \qquad (3.73) \\ &= \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

[6]The term "twist" has been used in different ways in the mechanisms and screw theory literature. In robotics, however, it is common to use the term to refer to a spatial velocity. We adopt this usage to minimize verbiage, e.g., "spatial velocity in the body frame" vs. "body twist."

[7]$se(3)$ is called the Lie algebra of the Lie group $SE(3)$. It consists of all possible $\dot{T}$ when $T = I$.

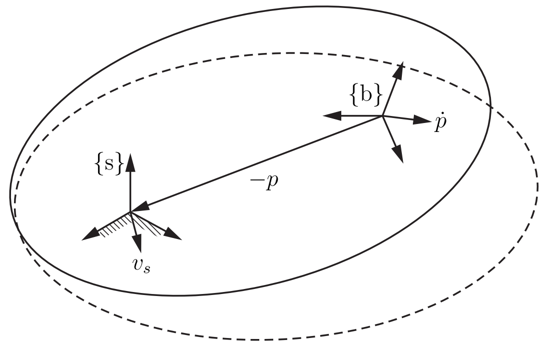


Figure 3.16: Physical interpretation of v_s. The initial (solid line) and displaced (dotted line) configurations of a rigid body.

Observe that the skew-symmetric matrix $[\omega_s] = \dot{R}R^T$ is the angular velocity expressed in fixed frame coordinates, but that $v_s = \dot{p} - \dot{R}R^T p$ is **not** the linear velocity of the body frame origin expressed in the fixed frame (that quantity would simply be $\dot{p}$). If we write v_s as

$$v_s = \dot{p} - \omega_s \times p = \dot{p} + \omega_s \times (-p), \tag{3.74}$$

the physical meaning of v_s can now be inferred: imagining an infinitely large moving body, v_s is the instantaneous velocity of the point on this body currently at the fixed frame origin, expressed in the fixed frame (see Figure 3.16).

As we did for ω_b and v_b, we assemble ω_s and v_s into a six-dimensional twist,

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6, \quad [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \dot{T}T^{-1} \in se(3), \tag{3.75}$$

where $[\mathcal{V}_s]$ is the 4×4 matrix representation of $\mathcal{V}_s$. We call $\mathcal{V}_s$ the **spatial velocity in the space frame**, or simply the **spatial twist**.

If we regard the moving body as being infinitely large, there is an appealing and natural symmetry between $\mathcal{V}_s = (\omega_s, v_s)$ and $\mathcal{V}_b = (\omega_b, v_b)$:

(i) ω_b is the angular velocity expressed in {b}, and ω_s is the angular velocity expressed in {s}; and

(ii) v_b is the linear velocity of a point at the origin of {b} expressed in {b}, and v_s is the linear velocity of a point at the origin of {s} expressed in {s}.

$\mathcal{V}_b$ can be obtained from $\mathcal{V}_s$ as follows:

$$\begin{aligned} [\mathcal{V}_b] &= T^{-1}\dot{T} \\ &= T^{-1}[\mathcal{V}_s]\,T. \end{aligned} \tag{3.76}$$

Going the other way,

$$[\mathcal{V}_s] = T\,[\mathcal{V}_b]\,T^{-1}. \tag{3.77}$$

Writing out the terms of Equation (3.77), we get

$$\mathcal{V}_s = \begin{bmatrix} R[\omega_b]R^T & -R[\omega_b]R^T p + Rv_b \\ 0 & 0 \end{bmatrix}$$

which, using $R[\omega]R^T = [R\omega]$ (Proposition 3.5) and $[\omega]p = -[p]\omega$ for $p, \omega \in \mathbb{R}^3$, can be manipulated into the following relation between $\mathcal{V}_b$ and $\mathcal{V}_s$:

$$\begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}.$$

Because the 6×6 matrix pre-multiplying $\mathcal{V}_b$ is useful for changing the frame of reference for twists and wrenches, as we will see shortly, we give it its own name.

Definition 3.6. Given $T = (R, p) \in SE(3)$, its **adjoint representation** $[\mathrm{Ad}_T]$ is

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}.$$

For any $\mathcal{V} \in \mathbb{R}^6$, the **adjoint map** associated with T is

$$\mathcal{V}' = [\mathrm{Ad}_T]\mathcal{V},$$

also sometimes written as

$$\mathcal{V}' = \mathrm{Ad}_T(\mathcal{V}).$$

In terms of the matrix form $[\mathcal{V}] \in se(3)$ of $\mathcal{V} \in \mathbb{R}^6$,

$$[\mathcal{V}'] = T[\mathcal{V}]T^{-1}.$$

The adjoint map satisfies the following properties, verifiable by direct calculation:

Proposition 3.14. *Let $T_1, T_2 \in SE(3)$, and $\mathcal{V} = (\omega, v)$. Then*

$$\mathrm{Ad}_{T_1}(\mathrm{Ad}_{T_2}(\mathcal{V})) = \mathrm{Ad}_{T_1T_2}(\mathcal{V}) \quad \textit{or} \quad [\mathrm{Ad}_{T_1}][\mathrm{Ad}_{T_2}]\mathcal{V} = [\mathrm{Ad}_{T_1T_2}]\mathcal{V}. \tag{3.78}$$

Also, for any $T \in SE(3)$ the following holds:

$$[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}], \tag{3.79}$$

The second property follows from the first by choosing $T_1 = T^{-1}$ and $T_2 = T$, so that

$$\mathrm{Ad}_{T^{-1}}(\mathrm{Ad}_T(\mathcal{V})) = \mathrm{Ad}_{T^{-1}T}(\mathcal{V}) = \mathrm{Ad}_I(\mathcal{V}) = \mathcal{V}. \tag{3.80}$$

3.3.2.1 Summary of Results on Twists

The main results on twists derived thus far are summarized in the following proposition:

Proposition 3.15. *Given a fixed (space) frame* $\{s\}$ *and a body frame* $\{b\}$*, let* $T_{sb}(t) \in SE(3)$ *be differentiable, where*

$$T_{sb}(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}. \tag{3.81}$$

Then

$$T_{sb}^{-1}\dot{T}_{sb} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3) \tag{3.82}$$

is the matrix representation of the **body twist***, and*

$$\dot{T}_{sb}T_{sb}^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3) \tag{3.83}$$

is the matrix representation of the **spatial twist***. The twists* $\mathcal{V}_s$ *and* $\mathcal{V}_b$ *are related by*

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b \tag{3.84}$$

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^T & 0 \\ -R^T[p] & R^T \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s. \tag{3.85}$$

Similarly, for any two frames {a} *and* {b}*, a twist represented as* $\mathcal{V}_a$ *in* {a} *is related to the representation* $\mathcal{V}_b$ *in* {b} *by*

$$\mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b, \;\; \mathcal{V}_b = [\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a.$$

Again analogous to angular velocities, it is important to realize that for a given twist, its fixed-frame representation $\mathcal{V}_s$ *does not depend on the choice of the body frame* {b}, and its body-frame representation $\mathcal{V}_b$ *does not depend on the choice of the fixed frame* {s}.

Example 3.3. Figure 3.17 shows a top view of a car with a single front wheel driving on a plane. The $\hat{\mathrm{z}}_{\mathrm{b}}$-axis of the body frame {b} is into the page and the $\hat{\mathrm{z}}_{\mathrm{s}}$-axis of the fixed frame {s} is out of the page. The angle of the front wheel of the car causes the car's motion to be a pure angular velocity w = 2 rad/s about an axis out of the page, at the point r in the plane. Inspecting the figure, we can write r as $r_s = (2, -1, 0)^T$ or $r_b = (2, -1.4, 0)^T$; w as $\omega_s = (0, 0, 2)^T$ or $\omega_b = (0, 0, -2)^T$; and T_{sb} as

$$T_{sb} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0.4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

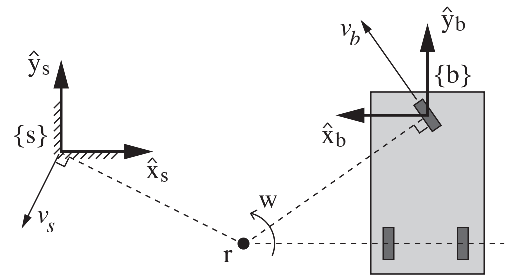


Figure 3.17: The twist corresponding to the instantaneous motion of the chassis of a three-wheeled vehicle can be visualized as an angular velocity w about the point r.

From the figure and simple geometry, we get

$$v_s = \omega_s \times (-r_s) = r_s \times \omega_s = (-2, -4, 0)^T$$
$$v_b = \omega_b \times (-r_b) = r_b \times \omega_b = (2.8, 4, 0)^T$$

to get the twists $\mathcal{V}_s$ and $\mathcal{V}_b$:

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \\ -4 \\ 0 \end{bmatrix}, \quad \mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 2.8 \\ 4 \\ 0 \end{bmatrix}.$$

To confirm these results, try calculating $\mathcal{V}_s = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b$.

3.3.2.2 The Screw Interpretation of a Twist

Just as an angular velocity ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis, a twist $\mathcal{V}$ can be interpreted as a **screw axis** $\mathcal{S}$ and a velocity $\dot{\theta}$ about the screw axis.

A screw axis represents the familiar motion of a screw: rotation about the axis while also translating along the axis. One representation of a screw axis $\mathcal{S}$ is the collection $\{q, \hat{s}, h\}$, where $q \in \mathbb{R}^3$ is any point on the axis; $\hat{s}$ is a unit vector in the direction of the axis; and h is the **screw pitch**, which defines the ratio of the linear velocity along the screw axis to the angular velocity $\dot{\theta}$ about the screw axis (Figure 3.18).

Using Figure 3.18 and geometry, we can write the twist $\mathcal{V} = (\omega, v)$ corresponding to an angular velocity $\dot{\theta}$ about $\mathcal{S}$ (represented by $\{q, \hat{s}, h\}$) as

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}.$$

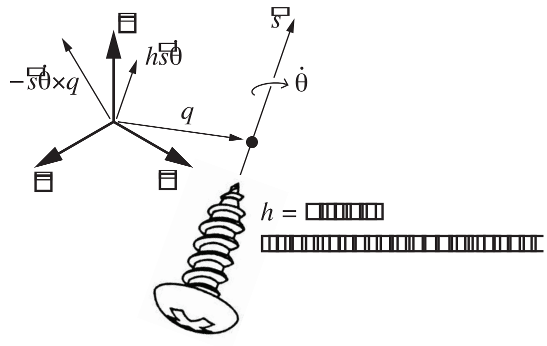


Figure 3.18: A screw axis $\mathcal{S}$ represented by a point q, a unit direction $\hat{s}$, and a pitch h.

Note that the linear velocity v is the sum of two terms: one due to translation along the screw axis, $h\hat{s}\dot{\theta}$, and one due to the linear motion at the origin induced by rotation about the axis, $-\hat{s}\dot{\theta} \times q$. The first term is in the direction of $\hat{s}$, while the second term is in the plane orthogonal to $\hat{s}$. It is not hard to show that, for any $\mathcal{V} = (\omega, v)$ where $\omega \neq 0$, there exists an equivalent screw axis $\{q, \hat{s}, h\}$ and velocity $\dot{\theta}$, where $\hat{s} = \omega/\|\omega\|$, $\dot{\theta} = \|\omega\|$, $h = \hat{\omega}^T v/\dot{\theta}$, and q is chosen so that the term $-\hat{s}\dot{\theta} \times q$ provides the portion of v orthogonal to the screw axis.

If $\omega = 0$, then the pitch h of the screw is infinite. So $\hat{s}$ is chosen as $v/\|v\|$, and $\dot{\theta}$ is interpreted as the linear velocity $\|v\|$ along $\hat{s}$.

Instead of representing the screw axis $\mathcal{S}$ using the cumbersome collection $\{q, \hat{s}, h\}$, with the possibility that h may be infinite and the non-uniqueness of q (any q along the screw axis may be used), we instead define the screw axis $\mathcal{S}$ using a normalized version of any twist $\mathcal{V} = (\omega, v)$ corresponding to motion along the screw:

(i) If $\omega \neq 0$: $\mathcal{S} = \mathcal{V}/\|\omega\| = (\omega/\|\omega\|, v/\|\omega\|)$. The screw axis $\mathcal{S}$ is simply $\mathcal{V}$ normalized by the length of the angular velocity vector. The angular velocity about the screw axis is $\dot{\theta} = \|\omega\|$, such that $\mathcal{S}\dot{\theta} = \mathcal{V}$.

(ii) If $\omega = 0$: $\mathcal{S} = \mathcal{V}/\|v\| = (0, v/\|v\|)$. The screw axis $\mathcal{S}$ is simply $\mathcal{V}$ normalized by the length of the linear velocity vector. The linear velocity along the screw axis is $\dot{\theta} = \|v\|$, such that $\mathcal{S}\dot{\theta} = \mathcal{V}$.

This leads to the following definition of a "unit" (normalized) screw axis:

Definition 3.7. For a given reference frame, a **screw axis** $\mathcal{S}$ is written

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6,$$

where either (i) $\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$. If (i) $\|\omega\| = 1$, then $v = -\omega \times q + h\omega$, where q is a point on the axis of the screw and h is the pitch of the screw ($h = 0$ for a pure rotation about the screw axis). If (ii) $\|\omega\| = 0$ and

$\|v\| = 1$, the pitch of the screw is $h = \infty$ and the twist is a translation along the axis defined by v.

The 4×4 matrix representation $[\mathcal{S}]$ of $\mathcal{S}$ is

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3), \quad [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3), \tag{3.86}$$

where the bottom row of $[\mathcal{S}]$ consists of all zeros.

Important: Although we use the pair (ω, v) for both normalized screw axes (where one of $\|\omega\|$ or $\|v\|$ must be unit) and general twists (where there are no constraints on ω and v), their meaning should be clear from context.

Since a screw axis $\mathcal{S}$ is just a normalized twist, a screw axis represented as $\mathcal{S}_a$ in a frame {a} is related to the representation $\mathcal{S}_b$ in a frame {b} by

$$\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \;\; \mathcal{S}_b = [\mathrm{Ad}_{T_{ba}}]\mathcal{S}_a.$$

3.3.3 Exponential Coordinate Representation of Rigid-Body Motions

3.3.3.1 Exponential Coordinates of Rigid-Body Motions

In the planar example in Section 3.1, we saw that any planar rigid-body displacement can be achieved by rotating the rigid body about some fixed point in the plane (for a pure translation, this point lies at infinity). A similar result also exists for spatial rigid-body displacements: called the **Chasles-Mozzi Theorem**, it states that every rigid-body displacement can be expressed as a displacement along a fixed screw axis $\mathcal{S}$ in space.

By analogy to the exponential coordinates $\hat{\omega}\theta$ for rotations, we define the six-dimensional **exponential coordinates of a homogeneous transformation** T as $\mathcal{S}\theta \in \mathbb{R}^6$, where $\mathcal{S}$ is the screw axis and θ is the distance that must be traveled along/about the screw axis to take a frame from the origin I to T. If the pitch of the screw axis $\mathcal{S} = (\omega, v)$ is finite, then $\|\omega\| = 1$ and θ corresponds to the angle of rotation about the screw axis. If the pitch of the screw is infinite, then $\omega = 0$ and $\|v\| = 1$, and θ corresponds to the linear distance traveled along the screw axis.

Also by analogy to the rotations, we define a matrix exponential and matrix logarithm:

$$\begin{array}{rlcl} \exp: & [\mathcal{S}]\theta \in se(3) & \rightarrow & T \in SE(3) \\ \log: & T \in SE(3) & \rightarrow & [\mathcal{S}]\theta \in se(3) \end{array}$$

We begin by deriving a closed-form expression for the matrix exponential $e^{[\mathcal{S}]\theta}$. Expanding the matrix exponential in series form leads to

$$\begin{aligned} e^{[\mathcal{S}]\theta} &= I + [\mathcal{S}]\theta + [\mathcal{S}]^2\frac{\theta^2}{2!} + [\mathcal{S}]^3\frac{\theta^3}{3!} + \ldots \\ &= \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \quad G(\theta) = I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \ldots \end{aligned} \tag{3.87}$$

Noting the similarity between $G(\theta)$ and the series definition for $e^{[\omega]\theta}$, it is tempting to write $I + G(\theta)[\omega] = e^{[\omega]\theta}$, and to conclude that $G(\theta) = (e^{[\omega]\theta} - I)[\omega]^{-1}$. This is wrong: $[\omega]^{-1}$ does not exist (try computing $\det[\omega]$).

Instead we make use of the result $[\omega]^3 = -[\omega]$ that was obtained from the Cayley-Hamilton Theorem. In this case $G(\theta)$ can be simplified to

$$\begin{aligned} G(\theta) &= I\theta + [\omega]\frac{\theta^2}{2!} + [\omega]^2\frac{\theta^3}{3!} + \ldots \\ &= I\theta + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \ldots\right)[\omega] + \left(\frac{\theta^3}{3!} - \frac{\theta^5}{5!} + \frac{\theta^7}{7!} - \ldots\right)[\omega]^2 \\ &= I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2. \end{aligned} \tag{3.88}$$

Putting everything together,

Proposition 3.16. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$, then for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.89}$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}. \tag{3.90}$$

The latter result of the proposition can be verified directly from the series expansion of $e^{[\mathcal{S}]\theta}$ with ω set to zero.

3.3.3.2 Matrix Logarithm of Rigid-Body Motions

The above derivation essentially provides a constructive proof of the Chasles-Mozzi Theorem. That is, given an arbitrary $(R, p) \in SE(3)$, one can always find a screw axis $\mathcal{S} = (\omega, v)$ and a scalar θ such that

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}. \tag{3.91}$$

In the simplest case, if $R = I$, then $\omega = 0$, and the preferred choice for v is $v = p/\|p\|$ (this makes $\theta = \|p\|$ the translation distance). If R is not the identity matrix and $\operatorname{tr} R \neq -1$, one solution is given by

$$[\omega] = \frac{1}{2\sin\theta}(R - R^T) \tag{3.92}$$

$$v = G^{-1}(\theta)p, \tag{3.93}$$

where θ satisfies $1 + 2\cos\theta = \operatorname{tr} R$. We leave as an exercise the verification of the following formula for $G^{-1}(\theta)$:

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2. \tag{3.94}$$

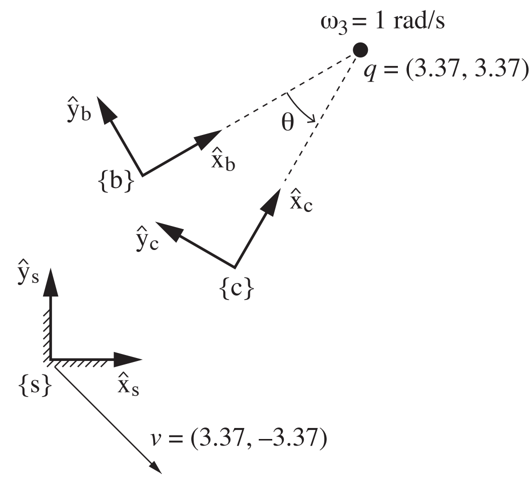


Figure 3.19: Two frames in a plane.

Finally, if $\text{tr } R = -1$, we choose $\theta = \pi$, and $[\omega]$ can be obtained via the matrix logarithm formula on $SO(3)$. Once $[\omega]$ and θ have been determined, v can then be obtained as $v = G^{-1}(\theta)p$.

Algorithm: Given (R, p) written as $T \in SE(3)$, find a $\theta \in [0, \pi]$ and a screw axis $\mathcal{S} = (\omega, v) \in \mathbb{R}^6$ such that $e^{[\mathcal{S}]\theta} = T$. The vector $\mathcal{S}\theta \in \mathbb{R}^6$ comprises the exponential coordinates for T and the matrix $[\mathcal{S}]\theta \in se(3)$ is a matrix logarithm of T.

(i) If $R = I$, then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$.

(ii) If $\text{tr } R = -1$, then set $\theta = \pi$, and $[\omega] = \log R$ as determined by the matrix logarithm formula on $SO(3)$ for the case $\text{tr } R = -1$. v is then given by $v = G^{-1}(\theta)p$.

(iii) Otherwise set $\theta = \cos^{-1}\left(\frac{\text{tr } R - 1}{2}\right) \in [0, \pi)$ and

$$[\omega] = \frac{1}{2\sin\theta}(R - R^T) \quad (3.95)$$

$$v = G^{-1}(\theta)p, \quad (3.96)$$

where $G^{-1}(\theta)$ is given by Equation (3.94).

Example 3.4. As an example, we consider the special case of planar rigid-body motions and examine the matrix logarithm formula on $SE(2)$. Suppose the initial and final configurations of the body are respectively represented by the $SE(2)$ matrices in Figure 3.19:

$$\begin{aligned} T_{sb} &= \begin{bmatrix} \cos 30^\circ & -\sin 30^\circ & 1 \\ \sin 30^\circ & \cos 30^\circ & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ T_{sc} &= \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 2 \\ \sin 60^\circ & \cos 60^\circ & 1 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

For this example, the rigid-body displacement occurs in the $\hat{\mathrm{x}}_\mathrm{s}$-$\hat{\mathrm{y}}_\mathrm{s}$ plane. The corresponding screw motion therefore has its screw axis in the direction of the $\hat{\mathrm{z}}_\mathrm{s}$-axis, and is of zero pitch. The screw axis $\mathcal{S} = (\omega, v)$, expressed in {s}, is of the form

$$\begin{aligned} \omega &= (0, 0, \omega_3)^T \\ v &= (v_1, v_2, 0)^T. \end{aligned}$$

Using this reduced form, we seek the screw motion that displaces the frame at T_{sb} to T_{sc}, i.e., $T_{sc} = e^{[\mathcal{S}]\theta} T_{sb}$, or

$$T_{sc} T_{sb}^{-1} = e^{[\mathcal{S}]\theta},$$

where

$$[\mathcal{S}] = \begin{bmatrix} 0 & -\omega_3 & v_1 \\ \omega_3 & 0 & v_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

We can apply the matrix logarithm algorithm directly to $T_{sc} T_{sb}^{-1}$ to obtain $[\mathcal{S}]$ (and therefore $\mathcal{S}$) and θ as follows:

$$[\mathcal{S}] = \begin{bmatrix} 0 & -1 & 3.37 \\ 1 & 0 & -3.37 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} \omega_3 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3.37 \\ -3.37 \end{bmatrix}, \quad \theta = \pi/6 \text{ rad (or } 30^\circ).$$

The value of $\mathcal{S}$ means that the constant screw axis, expressed in the fixed frame {s}, is represented by an angular velocity of 1 rad/s about $\hat{\mathrm{z}}_\mathrm{s}$ and a linear velocity of a point currently at the origin of {s} of $(3.37, -3.37)$, expressed in the {s} frame.

Alternatively, we can observe that the displacement is not a pure translation—T_{sb} and T_{sc} have rotation components that differ by an angle of 30°—and quickly determine that $\theta = 30^\circ$ and $\omega_3 = 1$. We can also graphically determine the point $q = (q_x, q_y)$ in the $\hat{\mathrm{x}}_\mathrm{s}$-$\hat{\mathrm{y}}_\mathrm{s}$ plane that the screw axis must pass through; for our example this point is given by $q = (3.37, 3.37)$.

3.4 Wrenches

Consider a linear force f acting on a rigid body at a point r. Defining a reference frame {a}, the point r can be represented as $r_a \in \mathbb{R}^3$, the force f can be

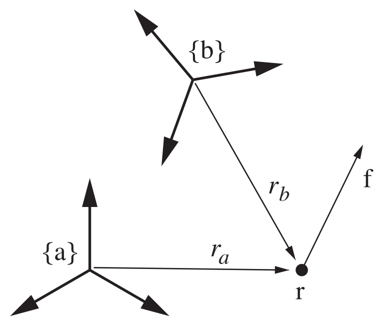


Figure 3.20: Relation between a wrench represented as $\mathcal{F}_a$ and $\mathcal{F}_b$.

represented as $f_a \in \mathbb{R}^3$, and this force creates a torque or **moment** $m_a \in \mathbb{R}^3$ in the {a} frame:

$$m_a = r_a \times f_a.$$

Note that the point of application of the force along the line of action of the force is immaterial.

Just as with twists, we can merge the moment and force into a single six-dimensional **spatial force**, or **wrench**, expressed in the {a} frame, $\mathcal{F}_a$:

$$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix} \in \mathbb{R}^6. \tag{3.97}$$

If more than one wrench acts on a rigid body, the total wrench on the body is simply the vector sum of the individual wrenches, provided the wrenches are expressed in the same frame. A wrench with zero linear component is called a **pure moment**.

A wrench in the {a} frame can be represented in another frame {b} if T_{ba} is known (Figure 3.20). One way to derive the relationship between $\mathcal{F}_a$ and $\mathcal{F}_b$ is to derive the appropriate transformations between the individual force and moment vectors based on techniques we have already used.

A simpler and more insightful way to derive the relationship between $\mathcal{F}_a$ and $\mathcal{F}_b$ is to (1) use the results we have already derived relating the representations $\mathcal{V}_a$ and $\mathcal{V}_b$ of the same twist, and (2) use the fact that the power generated (or dissipated) by an $(\mathcal{F}, \mathcal{V})$ pair must be the same regardless of the frame they are represented in. (Imagine if we could create power simply by changing our choice of a reference frame!) Recall that the dot product of a force and a velocity is power, and power is a coordinate-independent quantity. Because of this, we know

$$\mathcal{V}_b^T \mathcal{F}_b = \mathcal{V}_a^T \mathcal{F}_a. \tag{3.98}$$

From Proposition 3.15 we know that $\mathcal{V}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b$, and therefore Equation (3.98) can be rewritten as

$$\begin{aligned} \mathcal{V}_b^T \mathcal{F}_b &= ([\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b)^T \mathcal{F}_a \\ &= \mathcal{V}_b^T [\mathrm{Ad}_{T_{ab}}]^T \mathcal{F}_a. \end{aligned}$$

Since this must hold for all $\mathcal{V}_b$, this simplifies to

$$\mathcal{F}_b = [\mathrm{Ad}_{T_{ab}}]^T \mathcal{F}_a. \tag{3.99}$$

Similarly,

$$\mathcal{F}_a = [\mathrm{Ad}_{T_{ba}}]^T \mathcal{F}_b. \tag{3.100}$$

Proposition 3.17. *Given a wrench* F*, represented in* $\{a\}$ *as* $\mathcal{F}_a$ *and in* $\{b\}$ *as* $\mathcal{F}_b$*, the two representations are related by*

$$\begin{aligned} \mathcal{F}_b &= \mathrm{Ad}^T_{T_{ab}}(\mathcal{F}_a) = [\mathrm{Ad}_{T_{ab}}]^T \mathcal{F}_a && (3.101) \\ \mathcal{F}_a &= \mathrm{Ad}^T_{T_{ba}}(\mathcal{F}_b) = [\mathrm{Ad}_{T_{ba}}]^T \mathcal{F}_b. && (3.102) \end{aligned}$$

3.5 Summary

The following table succinctly summarizes some of the key concepts from the chapter, as well as the parallelism between rotations and rigid-body motions. For more details, consult the appropriate section of the chapter.

Rotations	**Rigid-Body Motions**
$R \in SO(3)$: 3×3 matrices satisfying $R^T R = I, \det R = 1$	$T \in SE(3)$: 4×4 matrices $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, where $R \in SO(3), p \in \mathbb{R}^3$
$R^{-1} = R^T$	$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$
change of coord frame: $R_{ab}R_{bc} = R_{ac}$, $R_{ab}p_b = p_a$	change of coord frame: $T_{ab}T_{bc} = T_{ac}$, $T_{ab}p_b = p_a$
rotating a frame {b}: $R = \text{Rot}(\hat{\omega}, \theta)$ $R_{sb'} = RR_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ $R_{sb''} = R_{sb}R$: rotate θ about $\hat{\omega}_b = \hat{\omega}$	displacing a frame {b}: $T = \begin{bmatrix} \text{Rot}(\hat{\omega}, \theta) & p \\ 0 & 1 \end{bmatrix}$ $T_{sb'} = TT_{sb}$: rotate θ about $\hat{\omega}_s = \hat{\omega}$ (moves {b} origin), translate p in {s} $T_{sb''} = T_{sb}T$: translate p in {b}, rotate θ about $\hat{\omega}$ in new body frame
unit rotation axis is $\hat{\omega} \in \mathbb{R}^3$, where $\\|\hat{\omega}\\| = 1$	"unit" screw axis is $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, where either (i) $\\|\omega\\| = 1$ or (ii) $\omega = 0$ and $\\|v\\| = 1$
	for a screw axis $\{q, \hat{s}, h\}$ with finite h, $\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s} \\ -\hat{s} \times q + h\hat{s} \end{bmatrix}$
angular velocity can be written $\omega = \hat{\omega}\dot{\theta}$	twist can be written $\mathcal{V} = \mathcal{S}\dot{\theta}$
for any 3-vector, e.g., $\omega \in \mathbb{R}^3$, $[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3)$ identities: for $\omega, x \in \mathbb{R}^3, R \in SO(3)$: $[\omega] = -[\omega]^T, [\omega]x = -[x]\omega$, $[\omega][x] = ([x][\omega])^T, R[\omega]R^T = [R\omega]$	for $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$, $[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$ (the pair (ω, v) can be a twist $\mathcal{V}$ or a "unit" screw axis $\mathcal{S}$, depending on the context)
$\dot{R}R^{-1} = [\omega_s]$, $R^{-1}\dot{R} = [\omega_b]$	$\dot{T}T^{-1} = [\mathcal{V}_s]$, $T^{-1}\dot{T} = [\mathcal{V}_b]$
	$[\text{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \in \mathbb{R}^{6\times 6}$ identities: $[\text{Ad}_T]^{-1} = [\text{Ad}_{T^{-1}}]$, $[\text{Ad}_{T_1}][\text{Ad}_{T_2}] = [\text{Ad}_{T_1 T_2}]$
change of coord frame: $\hat{\omega}_a = R_{ab}\hat{\omega}_b$, $\omega_a = R_{ab}\omega_b$	change of coord frame: $\mathcal{S}_a = [\text{Ad}_{T_{ab}}]\mathcal{S}_b$, $\mathcal{V}_a = [\text{Ad}_{T_{ab}}]\mathcal{V}_b$

continued...

Rotations (cont.)	**Rigid-Body Motions (cont.)**
$\hat{\omega}\theta \in \mathbb{R}^3$ are exp coords for $R \in SO(3)$	$\mathcal{S}\theta \in \mathbb{R}^6$ are exp coords for $T \in SE(3)$
$\exp : [\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3)$ $R = \mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} =$ $I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2$	$\exp : [\mathcal{S}]\theta \in se(3) \rightarrow T \in SE(3)$ $T = e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\hat{\omega}]\theta} & * \\ 0 & 1 \end{bmatrix}$ where $* =$ $(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2)v$
$\log : R \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3)$ algorithm in Section 3.2.3.3	$\log : T \in SE(3) \rightarrow [\mathcal{S}]\theta \in se(3)$ algorithm in Section 3.3.3.2
moment change of coord frame: $m_a = R_{ab} m_b$	wrench change of coord frame: $\mathcal{F}_a = (m_a, f_a) = [\mathrm{Ad}_{T_{ba}}]^T \mathcal{F}_b$

3.6 Software

The following functions are included in the software distribution accompanying the book. The code below is in MATLAB format, but the code is available in other languages. For more details on the software, consult the code and its documentation.

`invR = RotInv(R)`
Computes the inverse of the rotation matrix `R`.

`so3mat = VecToso3(omg)`
Returns the 3×3 skew-symmetric matrix corresponding to `omg`.

`omg = so3ToVec(so3mat)`
Returns the 3-vector corresponding to the 3×3 skew-symmetric matrix `so3mat`.

`[omghat,theta] = AxisAng3(expc3)`
Extracts the rotation axis $\hat{\omega}$ and rotation amount θ from the 3-vector $\hat{\omega}\theta$ of exponential coordinates for rotation, `expc3`.

`R = MatrixExp3(expc3)`
Computes the rotation matrix corresponding to a 3-vector of exponential coordinates for rotation. (Note: This function takes exponential coordinates as input, not an $so(3)$ matrix as implied by the function's name.)

`expc3 = MatrixLog3(R)`
Computes the 3-vector of exponential coordinates corresponding to the matrix logarithm of `R`. (Note: This function returns exponential coordinates, not an $so(3)$ matrix as implied by the function's name.)

`T = RpToTrans(R,p)`
Builds the homogeneous transformation matrix `T` corresponding to a rotation

matrix R $\in SO(3)$ and a position vector p $\in \mathbb{R}^3$.

`[R,p] = TransToRp(T)`
Extracts the rotation matrix and position vector from a homogeneous transformation matrix `T`.

`invT = TransInv(T)`
Computes the inverse of a homogeneous transformation matrix `T`.

`se3mat = VecTose3(V)`
Returns the $se(3)$ matrix corresponding to a 6-vector twist `V`.

`V = se3ToVec(se3mat)`
Returns the 6-vector twist corresponding to an $se(3)$ matrix `se3mat`.

`AdT = Adjoint(T)`
Computes the 6×6 adjoint representation $[\mathrm{Ad}_T]$ of the homogeneous transformation matrix `T`.

`S = ScrewToAxis(q,s,h)`
Returns a normalized screw axis representation $\mathcal{S}$ of a screw described by a unit vector `s` in the direction of the screw axis, located at the point `q`, with pitch `h`.

`[S,theta] = AxisAng(expc6)`
Extracts the normalized screw axis $\mathcal{S}$ and distance traveled along the screw θ from the 6-vector of exponential coordinates $\mathcal{S}\theta$.

`T = MatrixExp6(expc6)`
Computes the homogeneous transformation matrix corresponding to a 6-vector $\mathcal{S}\theta$ of exponential coordinates for rigid-body motion. (Note: This function takes exponential coordinates as input, not an $se(3)$ matrix as implied by the function's name.)

`expc6 = MatrixLog(T)`
Computes the 6-vector of exponential coordinates corresponding to the matrix logarithm of `T`. (Note: This function returns exponential coordinates, not an $se(3)$ matrix as implied by the function's name.)

3.7 Notes and References

More detailed coverage of the various parametrizations of $SO(3)$ can be found in, e.g., [96] and the references cited there. The treatment of the matrix exponential representation for screw motions is based on the work of Brockett [14]; a more mathematically detailed discussion can be found in [90]. Classical screw theory is presented in its original form in R. Ball's treatise [4]. More modern (algebraic and geometric) treatments can be found in, e.g., Bottema and Roth [13], Angeles [1], and McCarthy [86].

3.8 Exercises

1. In terms of the $\hat{\mathrm{x}}_{\mathrm{s}}$-$\hat{\mathrm{y}}_{\mathrm{s}}$-$\hat{\mathrm{z}}_{\mathrm{s}}$ coordinates of a fixed space frame {s}, the frame {a} has an $\hat{\mathrm{x}}_{\mathrm{a}}$-axis pointing in the direction $(0, 0, 1)$ and a $\hat{\mathrm{y}}_{\mathrm{a}}$-axis pointing in the direction $(-1, 0, 0)$, and the frame {b} has an $\hat{\mathrm{x}}_{\mathrm{b}}$-axis pointing in the direction $(1, 0, 0)$ and a $\hat{\mathrm{y}}_{\mathrm{b}}$-axis pointing in the direction $(0, 0, -1)$.
(a) Give your best hand drawing of the three frames. Draw them at different locations so they are easy to see.
(b) Write the rotation matrices R_{sa} and R_{sb}.
(c) Given R_{sb}, how do you calculate R_{sb}^{-1} without using a matrix inverse? Write R_{sb}^{-1} and verify its correctness with your drawing.
(d) Given R_{sa} and R_{sb}, how do you calculate R_{ab} (again no matrix inverses)? Compute the answer and verify its correctness with your drawing.
(e) Let $R = R_{sb}$ be considered as a transformation operator consisting of a rotation about $\hat{\mathrm{x}}$ by $-90°$. Calculate $R_1 = R_{sa}R$, and think of R_{sa} as a representation of an orientation, R as a rotation of R_{sa}, and R_1 as the new orientation after performing the rotation. Does the new orientation R_1 correspond to rotating R_{sa} by $-90°$ about the world-fixed $\hat{\mathrm{x}}_{\mathrm{s}}$-axis or the body-fixed $\hat{\mathrm{x}}_{\mathrm{a}}$-axis? Now calculate $R_2 = RR_{sa}$. Does the new orientation R_2 correspond to rotating R_{sa} by $-90°$ about the world-fixed $\hat{\mathrm{x}}_{\mathrm{s}}$-axis or the body-fixed $\hat{\mathrm{x}}_{\mathrm{a}}$-axis?
(f) Use R_{sb} to change the representation of the point $p_b = (1, 2, 3)^T$ (in {b} coordinates) to {s} coordinates.
(g) Choose a point p represented by $p_s = (1, 2, 3)^T$ in {s} coordinates. Calculate $p' = R_{sb}p_s$ and $p'' = R_{sb}^T p_s$. For each operation, should the result be interpreted as changing coordinates (from the {s} frame to {b}) without moving the point p, or as moving the location of the point without changing the reference frame of the representation?
(h) An angular velocity w is represented in {s} as $\omega_s = (3, 2, 1)^T$. What is its representation ω_a?
(h) By hand, calculate the matrix logarithm $[\hat{\omega}]\theta$ of R_{sa}. (You may verify your answer with software.) Extract the unit angular velocity $\hat{\omega}$ and rotation amount θ. Redraw the fixed frame {s} and in it draw $\hat{\omega}$.
(i) Calculate the matrix exponential corresponding to the exponential coordinates of rotation $\hat{\omega}\theta = (1, 2, 0)^T$. Draw the corresponding frame relative to {s}, as well as the rotation axis $\hat{\omega}$.

2. Let p be a point whose coordinates are $p = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}})$ with respect to the fixed frame $\hat{\mathrm{x}}$-$\hat{\mathrm{y}}$-$\hat{\mathrm{z}}$. Suppose p is rotated about the fixed frame $\hat{\mathrm{x}}$-axis by 30 degrees, then about the fixed frame $\hat{\mathrm{y}}$-axis by 135 degrees, and finally about the fixed frame $\hat{\mathrm{z}}$-axis by -120 degrees. Denote the coordinates of this newly rotated point by p'.
(a) What are the coordinates of p'?

(b) Find the rotation matrix R such that $p' = Rp$ for the p' you obtained in (a).

3. Suppose $p_i \in \mathbb{R}^3$ and $p_i' \in \mathbb{R}^3$ are related by $p_i' = Rp_i$, $i = 1, 2, 3$, for some unknown rotation matrix R to be determined. Find, if it exists, the rotation R for the three input-output pairs $p_i \mapsto p_i'$:

$$\begin{aligned} p_1 = (\sqrt{2}, 0, 2)^T \quad &\mapsto \quad p_1' = (0, 2, \sqrt{2})^T \\ p_2 = (1, 1, -1)^T \quad &\mapsto \quad p_2' = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\sqrt{2})^T \\ p_3 = (0, 2\sqrt{2}, 0)^T \quad &\mapsto \quad p_3' = (-\sqrt{2}, \sqrt{2}, -2)^T. \end{aligned}$$

4. In this exercise you are asked to prove the property $R_{ab}R_{bc} = R_{ac}$ of Equation (3.22). Define the unit axes of frames {a}, {b}, and {c} by the triplet of orthogonal unit vectors $\{\hat{\mathrm{x}}_\mathrm{a}, \hat{\mathrm{y}}_\mathrm{a}, \hat{\mathrm{z}}_\mathrm{a}\}$, $\{\hat{\mathrm{x}}_\mathrm{b}, \hat{\mathrm{y}}_\mathrm{b}, \hat{\mathrm{z}}_\mathrm{b}\}$, and $\{\hat{\mathrm{x}}_\mathrm{c}, \hat{\mathrm{y}}_\mathrm{c}, \hat{\mathrm{z}}_\mathrm{c}\}$, respectively. Suppose that the unit axes of frame {b} can be expressed in terms of the unit axes of frame {a} by

$$\begin{aligned} \hat{\mathrm{x}}_\mathrm{b} &= r_{11}\hat{\mathrm{x}}_\mathrm{a} + r_{21}\hat{\mathrm{y}}_\mathrm{a} + r_{31}\hat{\mathrm{z}}_\mathrm{a} \\ \hat{\mathrm{y}}_\mathrm{b} &= r_{12}\hat{\mathrm{x}}_\mathrm{a} + r_{22}\hat{\mathrm{y}}_\mathrm{a} + r_{32}\hat{\mathrm{z}}_\mathrm{a} \\ \hat{\mathrm{z}}_\mathrm{b} &= r_{13}\hat{\mathrm{x}}_\mathrm{a} + r_{23}\hat{\mathrm{y}}_\mathrm{a} + r_{33}\hat{\mathrm{z}}_\mathrm{a}. \end{aligned}$$

Similarly, suppose the unit axes of frame $\{c\}$ can be expressed in terms of the unit axes of frame $\{b\}$ by

$$\begin{aligned} \hat{\mathrm{x}}_\mathrm{c} &= s_{11}\hat{\mathrm{x}}_\mathrm{b} + s_{21}\hat{\mathrm{y}}_\mathrm{b} + s_{31}\hat{\mathrm{z}}_\mathrm{b} \\ \hat{\mathrm{y}}_\mathrm{c} &= s_{12}\hat{\mathrm{x}}_\mathrm{b} + s_{22}\hat{\mathrm{y}}_\mathrm{b} + s_{32}\hat{\mathrm{z}}_\mathrm{b} \\ \hat{\mathrm{z}}_\mathrm{c} &= s_{13}\hat{\mathrm{x}}_\mathrm{b} + s_{23}\hat{\mathrm{y}}_\mathrm{b} + s_{33}\hat{\mathrm{z}}_\mathrm{b}. \end{aligned}$$

From the above prove that $R_{ab}R_{bc} = R_{ac}$.

5. Find the exponential coordinates $\hat{\omega}\theta \in \mathbb{R}^3$ for the $SO(3)$ matrix

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}.$$

6. Given $R = \mathrm{Rot}(\hat{\mathrm{x}}, \frac{\pi}{2})\mathrm{Rot}(\hat{\mathrm{z}}, \pi)$, find the unit vector $\hat{\omega}$ and angle θ such that $R = e^{[\hat{\omega}]\theta}$.

7. (a) Given the rotation matrix

$$R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

find all possible values for $\hat{\omega} \in \mathbb{R}^3, \|\hat{\omega}\| = 1$, and $\theta \in [0, 2\pi]$ such that $e^{[\hat{\omega}]\theta} = R$.
(b) The two vectors $v_1, v_2 \in \mathbb{R}^3$ are related by

$$v_2 = Rv_1 = e^{[\hat{\omega}]\theta} v_1$$

where $\hat{\omega} \in \mathbb{R}^3$ has length one, and $\theta \in [-\pi, \pi]$. Given $\hat{\omega} = (\frac{2}{3}, \frac{2}{3}, \frac{1}{3})^T, v_1 = (1, 0, 1)^T, v_2 = (0, 1, 1)^T$, find all angles θ that satisfy the above equation.

8. (a) Suppose we seek the logarithm of a rotation matrix R whose trace is -1. From the exponential formula

$$e^{[\hat{\omega}]\theta} = I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2, \quad \|\omega\| = 1,$$

and recalling that $\operatorname{tr} R = -1$ implies $\theta = \pi$, the above equation simplifies to

$$R = I + 2[\hat{\omega}]^2 = \begin{bmatrix} 1 - 2(\hat{\omega}_2^2 + \hat{\omega}_3^2) & 2\hat{\omega}_1\hat{\omega}_2 & 2\hat{\omega}_1\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_2 & 1 - 2(\hat{\omega}_1^2 + \hat{\omega}_3^2) & 2\hat{\omega}_2\hat{\omega}_3 \\ 2\hat{\omega}_1\hat{\omega}_2 & 2\hat{\omega}_2\hat{\omega}_3 & 1 - 2(\hat{\omega}_1^2 + \hat{\omega}_2^2) \end{bmatrix}$$

Using the fact that $\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$, is it correct to conclude that

$$\hat{\omega}_1 = \sqrt{\frac{r_{11} + 1}{2}}, \qquad \hat{\omega}_2 = \sqrt{\frac{r_{22} + 1}{2}}, \qquad \hat{\omega}_3 = \sqrt{\frac{r_{33} + 1}{2}}$$

is also a solution?
(c) Using the fact that $[\hat{\omega}]^3 = -[\hat{\omega}]$, the identity $R = I + 2[\hat{\omega}]^2$ can also be written in the alternative form

$$\begin{aligned} R - I &= 2[\hat{\omega}]^2 \\ [\hat{\omega}]\,(R - I) &= 2\,[\hat{\omega}]^3 = -2\,[\hat{\omega}] \\ [\hat{\omega}]\,(R + I) &= 0. \end{aligned}$$

The resulting equation is a system of three linear equations in $(\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$. What is the relation between the solution to this linear system and the logarithm of R?

9. Exploiting all of the known properties of rotation matrices, determine the minimum number of arithmetic operations (multiplication and division, addition and subtraction) required to multiply two rotation matrices.

10. Due to finite arithmetic precision, the numerically obtained product of two rotation matrices is not necessarily a rotation matrix; that is, the resulting rotation A may not exactly satisfy $A^T A = I$ as desired. Devise an iterative numerical procedure that takes an arbitrary matrix $A \in \mathbb{R}^{3\times 3}$, and produces a matrix $R \in SO(3)$ that minimizes

$$\|A - R\|^2 = \operatorname{tr}\,(A - R)(A - R)^T.$$

11. (a) If $A = [a]$ and $B = [b]$ for $a, b \in \mathbb{R}^3$, then under what conditions on a and b does $e^A e^B = e^{A+B}$?
(b) If $A = [\mathcal{V}_a]$ and $B = [\mathcal{V}_b]$, where $\mathcal{V}_a = (\omega_a, v_a)$ and $\mathcal{V}_b = (\omega_b, v_b)$ are arbitrary twists, then under what conditions on $\mathcal{V}_a$ and $\mathcal{V}_b$ does $e^A e^B = e^{A+B}$? Try to give a physical description of this condition.
(c) Under what conditions on general $A, B \in \mathbb{R}^{n\times n}$ does $e^A e^B = e^{A+B}$?

12. (a) Given a rotation matrix $A = \mathrm{Rot}(\hat{z}, \alpha)$, where $\mathrm{Rot}(\hat{z}, \alpha)$ indicates a rotation about the $\hat{z}$-axis by an angle α, find all rotation matrices $R \in SO(3)$ that satisfy $AR = RA$.
(b) Given rotation matrices $A = \mathrm{Rot}(\hat{z}, \alpha)$ and $B = \mathrm{Rot}(\hat{z}, \beta)$, with $\alpha \neq \beta$, find all rotation matrices $R \in SO(3)$ that satisfy $AR = RB$.
(c) Given arbitrary rotation matrices $A, B \in SO(3)$, find all solutions $R \in SO(3)$ to the equation $AR = RB$.

13. (a) Show that the three eigenvalues of a rotation matrix $R \in SO(3)$ each have unit magnitude, and conclude that they can always be written $\{\mu + i\nu, \mu - i\nu, 1\}$, where $\mu^2 + \nu^2 = 1$.
(b) Show that a rotation matrix $R \in SO(3)$ can always be factored in the form

$$R = A \begin{bmatrix} \mu & \nu & 0 \\ -\nu & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1},$$

where $A \in SO(3)$ and $\mu^2 + \nu^2 = 1$. (*Hint*: Denote the eigenvector associated with the eigenvalue $\mu + i\nu$ by $x + iy$, $x, y \in \mathbb{R}^3$, and the eigenvector associated with the eigenvalue 1 by $z \in \mathbb{R}^3$. For the purposes of this problem you may assume that the set of vectors $\{x, y, z\}$ can always be chosen to be linearly independent.)

14. Given $\omega \in \mathbb{R}^3$, $\|\omega\| = 1$, and θ a nonzero scalar, show that

$$\left(I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right)^{-1} = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2.$$

Hint: From the identity $[\omega]^3 = -[\omega]$, express the inverse expressed as a quadratic matrix polynomial in $[\omega]$.

15. (a) Given a fixed frame $\{0\}$, and a moving frame $\{1\}$ in the identity orientation, perform the following sequence of rotations on $\{1\}$:

(i) Rotate $\{1\}$ about the $\{0\}$ frame x-axis by α; call this new frame $\{2\}$.

(ii) Rotate $\{2\}$ about the $\{0\}$ frame y-axis by β; call this new frame $\{3\}$.

(iii) Rotate {3} about the {0} frame z-axis by γ; call this new frame {4}.

What is the final orientation R_{04}?
(b) Suppose that the third step in (a) is replaced by the following: "Rotate {3} about the frame {3} z-axis by γ; call this new frame {4}." What is the final orientation R_{04}?
(c) From the given transformations, find T_{ca}:

$$T_{ab} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & -1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{cb} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

16. In terms of the $\hat{\mathrm{x}}_\mathrm{s}$-$\hat{\mathrm{y}}_\mathrm{s}$-$\hat{\mathrm{z}}_\mathrm{s}$ coordinates of a fixed space frame {s}, the frame {a} has an $\hat{\mathrm{x}}_\mathrm{a}$-axis pointing in the direction $(0, 0, 1)$ and a $\hat{\mathrm{y}}_\mathrm{a}$-axis pointing in the direction $(-1, 0, 0)$, and the frame {b} has an $\hat{\mathrm{x}}_\mathrm{b}$-axis pointing in the direction $(1, 0, 0)$ and a $\hat{\mathrm{y}}_\mathrm{b}$-axis pointing in the direction $(0, 0, -1)$. The origin of {a} is at (3,0,0) in {s} and the origin of {b} is at (0,2,0).
(a) Give your best hand drawing showing {a} and {b} relative to {s}.
(b) Write the rotation matrices R_{sa} and R_{sb} and the transformation matrices T_{sa} and T_{sb}.
(c) Given T_{sb}, how do you calculate T_{sb}^{-1} without using a matrix inverse? Write T_{sb}^{-1} and verify its correctness with your drawing.
(d) Given T_{sa} and T_{sb}, how do you calculate T_{ab} (again no matrix inverses)? Compute the answer and verify its correctness with your drawing.
(e) Let $T = T_{sb}$ be considered as a transformation operator consisting of a rotation about $\hat{\mathrm{x}}$ by $-90°$ and a translation along $\hat{\mathrm{y}}$ by 2 units. Calculate $T_1 = T_{sa}T$. Does T_1 correspond to a rotation and translation about $\hat{\mathrm{x}}_\mathrm{s}$ and $\hat{\mathrm{y}}_\mathrm{s}$, respectively (world-fixed transformation of T_{sa}), or a rotation and translation about $\hat{\mathrm{x}}_\mathrm{a}$ and $\hat{\mathrm{y}}_\mathrm{a}$, respectively (body-fixed transformation of T_{sa})? Now calculate $T_2 = TT_{sa}$. Does T_2 correspond to a body-fixed or world-fixed transformation of T_{sa}?
(f) Use T_{sb} to change the representation of the point $p_b = (1, 2, 3)^T$ (in {b} coordinates) to {s} coordinates.
(g) Choose a point p represented by $p_s = (1, 2, 3)^T$ in {s} coordinates. Calculate $p' = T_{sb}p_s$ and $p'' = T_{sb}^{-1}p_s$. For each operation, should the result be interpreted as changing coordinates (from the {s} frame to {b}) without moving the point p, or as moving the location of the point without changing the reference frame of the representation?
(g) A twist V is represented in {s} as $\mathcal{V}_s = (3, 2, 1, -1, -2, -3)^T$. What is its representation $\mathcal{V}_a$?
(h) By hand, calculate the matrix logarithm $[\mathcal{S}]\theta$ of T_{sa}. (You may verify your answer with software.) Extract the normalized screw axis $\mathcal{S}$ and rotation amount θ. Get the $\{q, \hat{s}, h\}$ representation of the screw axis. Redraw the fixed frame {s} and in it draw $\mathcal{S}$.
(i) Calculate the matrix exponential corresponding to the exponential coordi-

nates of rigid-body motion $\mathcal{S}\theta = (0, 1, 2, 3, 0, 0)^T$. Draw the corresponding frame relative to {s}, as well as the screw axis $\mathcal{S}$.

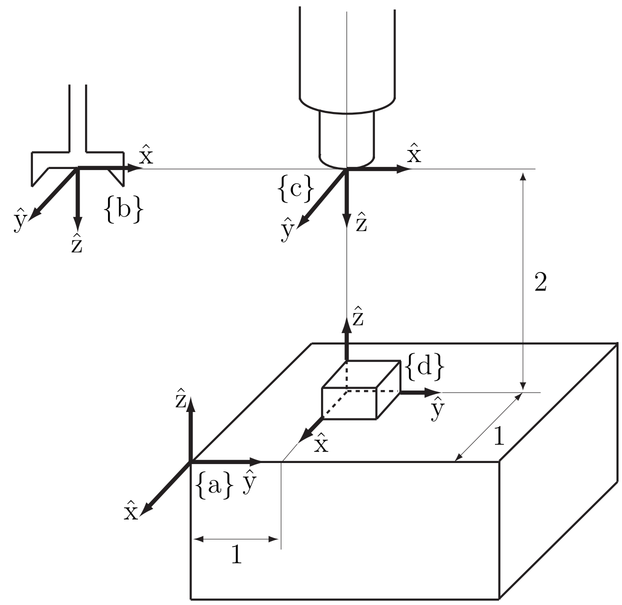


Figure 3.21: Four reference frames defined in a robot's workspace.

17. Four reference frames are shown in the robot workspace of Figure 3.21: the fixed frame {a}, the end-effector frame effector {b}, camera frame {c}, and workpiece frame {d}.
(a) Find T_{ad} and T_{cd} in terms of the dimensions given in the figure.
(b) Find T_{ab} given that

$$T_{bc} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

18. Consider a robot arm mounted on a spacecraft as shown in Figure 3.22, in which frames are attached to the earth {e}, satellite {s}, the spacecraft {a}, and the robot arm {r}, respectively.
(a) Given T_{ea}, T_{ar}, and T_{es}, find T_{rs}.
(b) Suppose the frame {s} origin as seen from {e} is $(1, 1, 1)$, and

$$T_{er} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

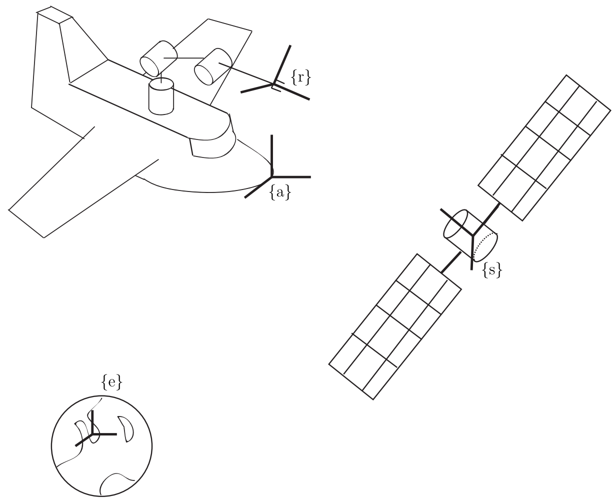


Figure 3.22: A robot arm mounted on a spacecraft.

Write down the coordinates of the frame {s} origin as seen from frame {r}.

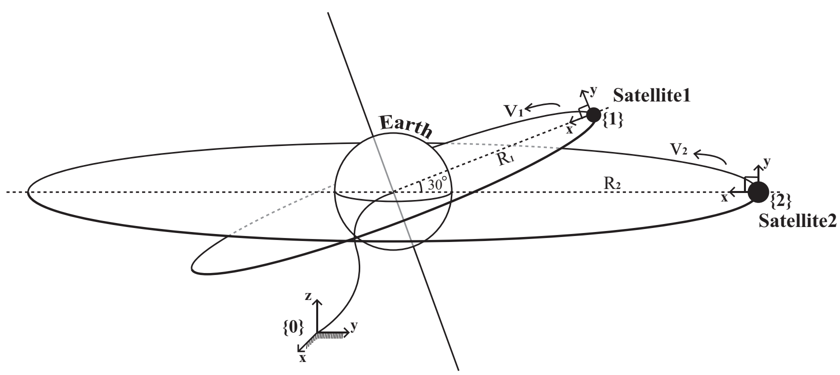


Figure 3.23: Two satellites circling the earth.

19. Two satellites are circling the earth as shown in Figure 3.23. Frames {1} and {2} are rigidly attached to the satellites such that their x̂-axes always point toward the earth. Satellite 1 moves at a constant speed v_1, while satellite 2 moves at a constant speed v_2. To simplify matters assume the earth is does not rotate about its own axis. The fixed frame {0} is located at the center of the earth. Figure 3.23 shows the position of the two satellites at $t = 0$.
(a) Derive frames T_{01}, T_{02} as a function of t.
(b) Using your results obtain in (a), find T_{21} as a function of t.

20. Consider the high-wheel bicycle of Figure 3.24, in which the diameter of the

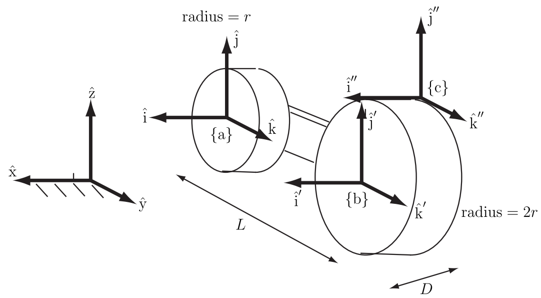


Figure 3.24: A high-wheel bicycle.

front wheel is twice that of the rear wheel. Frames {a} and {b} are attached to the centers of each wheel, and frame {c} is attached to the top of the front wheel. Assuming the bike moves forward in the $\hat{y}$ direction, find T_{ac} as a function of the front wheel's rotation angle θ (assume $\theta = 0$ at the instant shown in the figure).

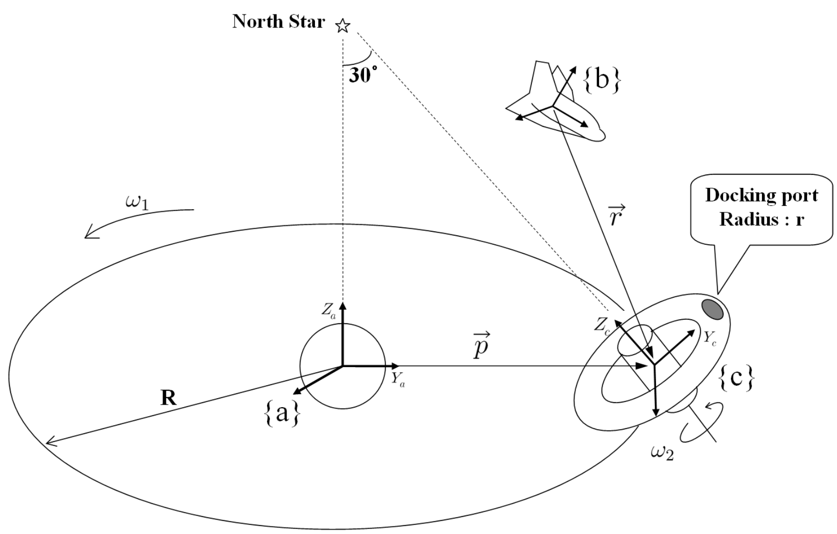


Figure 3.25: Spacecraft and space station.

21. The space station of Figure 3.25 moves in circular orbit around the earth, and at the same time rotates about an axis always pointing toward the north star. Due to an instrument malfunction, a spacecraft heading toward the space station is unable to locate the docking port. An earth-based ground station sends the following information to the spacecraft:

$$T_{ab} = \begin{bmatrix} 0 & -1 & 0 & -100 \\ 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 500 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad p_a = \begin{bmatrix} 0 \\ 800 \\ 0 \end{bmatrix},$$

where p_a is the vector $\vec{p}$ expressed in {a} frame coordinates.
(a) From the given information, find r_b, the vector $\vec{r}$ expressed in {b} frame coordinates.
(b) Determine T_{bc} at the instant shown in the figure. Assume here that the $\hat{y}$ and $\hat{z}$ axes of the {a} and {c} frames are coplanar with the docking port.

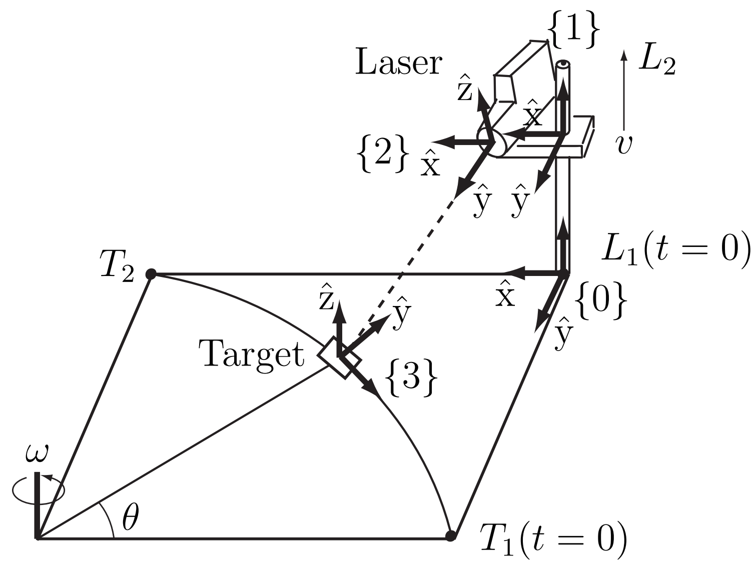


Figure 3.26: A laser tracking a moving target.

22. A target moves along a circular path at constant angular velocity ω rad/s as shown in Figure 3.26. The target is tracked by a laser mounted on a moving platform, rising vertically at constant speed v. Assume the laser and the platform start at L_1 at $t = 0$, while the target starts at frame T_1.
(a) Derive frames T_{01}, T_{12}, T_{03} as a function of t.
(b) Using your results from part (a), derive T_{23} as a function of t.

23. Two toy cars are moving on a round table as shown in Figure 3.27. Car 1 moves at a constant speed v_1 along the circumference of the table, while car 2 moves at a constant speed v_2 along a radius; the positions of the two vehicles at $t = 0$ are shown in the figure.
(a) Find T_{01}, T_{02} as a function of t.
(b) Find T_{12} as a function of t.

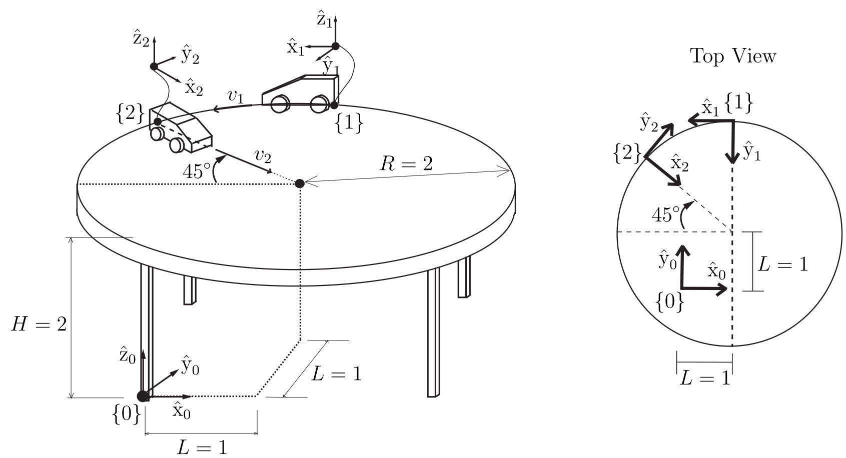


Figure 3.27: Two toy cars on a round table.

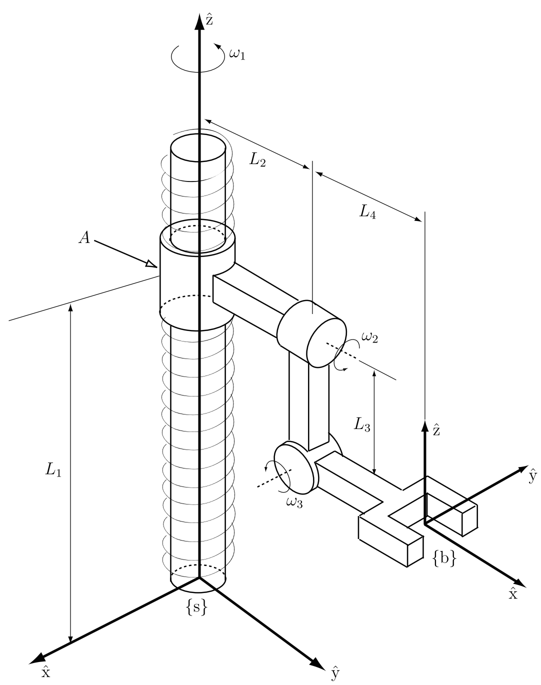


Figure 3.28: A robot arm with a screw joint.

24. Figure 3.28 shows the configuration, at $t = 0$, of a robot arm whose first joint is a screw joint of pitch $h = 2$. The arm's link lengths are $L_1 = 10$, $L_2 = L_3 = 5$, and $L_4 = 3$. Suppose all joint angular velocities are constant, with values $\omega_1 = \frac{\pi}{4}$, $\omega_2 = \frac{\pi}{8}$, $w_3 = -\frac{\pi}{4}$ rad/s. Find $T_{sb}(4) \in SE(3)$, i.e., the end-effector frame $\{b\} \in SE(3)$ relative to the fixed frame $\{s\}$, at time $t = 4$.

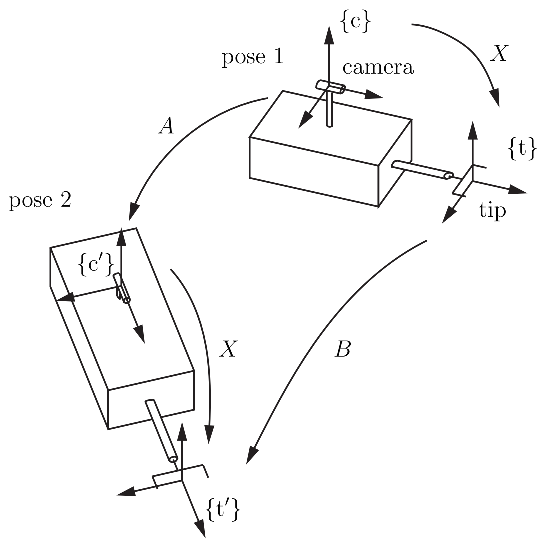


Figure 3.29: A camera rigidly attached to a robot arm.

25. A camera is rigidly attached to a robot arm as shown in Figure 3.29. The transformation $X \in SE(3)$ is constant. The robot arm moves from pose 1 to pose 2. The transformations $A \in SE(3)$ and $B \in SE(3)$ are measured and assumed known.
(a) Suppose X and A are given as follows:

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is B?
(b) Now suppose

$$A = \begin{bmatrix} R_A & p_A \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} R_B & p_B \\ 0 & 1 \end{bmatrix}$$

are known and we wish to find

$$X = \begin{bmatrix} R_X & p_X \\ 0 & 1 \end{bmatrix}.$$

Suppose $R_A = e^{[\alpha]}$ and $R_B = e^{[\beta]}$. What are the conditions on $\alpha \in \mathbb{R}^3$ and $\beta \in \mathbb{R}^3$ for a solution R_X to exist?

(c) Now suppose we have a set of k equations

$$A_i X = X B_i \text{ for } i = 1, \ldots, k$$

Assume A_i and B_i are all known. What is the minimum number of k for which a unique solution exists?

26. Draw the screw axis with $q = (3, 0, 0)^T$, $\hat{s} = (0, 0, 1)^T$, and $h = 2$.

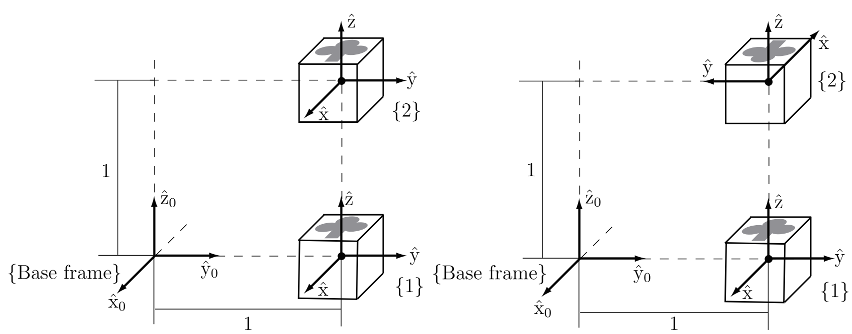
27. Draw the screw axis corresponding to the twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)^T$.

28. Assume that the space frame angular velocity is $\omega_s = (1, 2, 3)^T$ for a moving body with frame {b} at

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

in the world frame {s}. Calculate the angular velocity ω_b in {b}.

29. Two frames {a} and {b} are attached to a moving rigid body. Show that the twist of {a} in space frame coordinates is the same as the twist of {b} in space frame coordinates.



(a) a (b) b

Figure 3.30: A cube undergoing two different screw motions.

30. A cube undergoes two different screw motions from frame {1} to frame {2} as shown in Figure 3.30. In both cases (a) and (b), the initial configuration of the cube is

$$T_{01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) For each case (a) and (b), find the exponential coordinates $\mathcal{S}\theta = (\omega, v)$ such that $T_{02} = e^{[\mathcal{S}]\theta} T_{01}$, where no constraints are placed on ω or v.
(b) Repeat (a), this time with the constraint that $\|\omega\| \in [-\pi, \pi]$.

31. In Example 3.2 and Figure 3.15, the block the robot must pick up weighs 1 kg, which means the robot must provide approximately 10 N of force in the $\hat{\mathrm{z}}_{\mathrm{e}}$ direction of the block's frame {e} (which you can assume is at the block's center of mass). Express this force as a wrench in the {e} frame, $\mathcal{F}_e$. Given the transformation matrices in Example 3.2, express this same wrench in the end-effector frame {c} as $\mathcal{F}_c$.

32. Given two reference frames {a} and {b} in physical space, and a fixed frame {o}, define the distance between frames {a} and {b} as

$$\mathrm{dist}(T_{oa}, T_{ob}) \equiv \sqrt{\theta^2 + ||p_{ab}||^2}$$

where $R_{ab} = e^{[\hat{\omega}]\theta}$. Suppose the fixed frame is displaced to another frame {o′}, and that $T_{o'a} = ST_{oa}$, $T_{o'b} = ST_{o'b}$ for some constant $S = (R_s, p_s) \in SE(3)$.
(a) Evaluate $\mathrm{dist}(T_{o'a}, T_{o'b})$ using the above distance formula.
(b) Under what conditions on S does $\mathrm{dist}(T_{oa}, T_{ob}) = \mathrm{dist}(T_{o'a}, T_{o'b})$?

33. (a) Find the general solution to the differential equation $\dot{x} = Ax$, where

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}.$$

What happens to the solution $x(t)$ as $t \to \infty$?
(b) Do the same for

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

What happens to the solution $x(t)$ as $t \to \infty$?

34. Let $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times 2}$, and consider the linear differential equation $\dot{x}(t) = Ax(t)$. Suppose that

$$x(t) = \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$$

is a solution for the initial condition $x(0) = (1, -3)$, and

$$x(t) = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}.$$

is a solution for the initial condition $x(0) = (1, 1)$. Find A and e^{At}.

35. Given a differential equation of the form $\dot{x} = Ax + f(t)$, where $x \in \mathbb{R}^n$ and $f(t)$ is a given differentiable function of t. Show that the general solution can be written

$$x(t) = e^{At}x(0) = \int_0^t e^{A(t-s)} f(s)\, ds.$$

(*Hint*: Define $z(t) = e^{-At}x(t)$, and evaluate $\dot{z}(t)$.)

36. Referring to Appendix B, answer the following questions related to ZXZ Euler angles.
(a) Derive a procedure for finding the ZXZ Euler angles of a rotation matrix.
(b) Using the results of (a), find the ZXZ Euler angles for the following rotation matrix:

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

37. Consider a wrist mechanism with two revolute joints θ_1 and θ_2, in which the end-effector frame orientation $R \in SO(3)$ is given by

$$R = e^{[\hat{\omega}_1]\theta_1} e^{[\hat{\omega}_2]\theta_2},$$

with $\hat{\omega}_1 = (0,0,1)$ and $\hat{\omega}_2 = (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Determine whether the following orientation is reachable (that is, find, if it exists, a solution (θ_1, θ_2) for the following R):

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

38. Show that rotation matrices of the form

$$\begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

can be parametrized using just two parameters θ and ϕ as follows:

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \end{bmatrix}.$$

What should the range of values be for θ and ϕ?

39. Figure 3.31 shows a three degree of freedom wrist mechanism in its zero position (that is, with all its joints set to zero).
(a) Express the tool frame orientation $R_{03} = R(\alpha, \beta, \gamma)$ as a product of three rotation matrices.
(b) Find all possible angles (α, β, γ) for the two values of R_{03} given below. If no solution exists, explain why in terms of the analogy between $SO(3)$ and the solid ball of radius π.

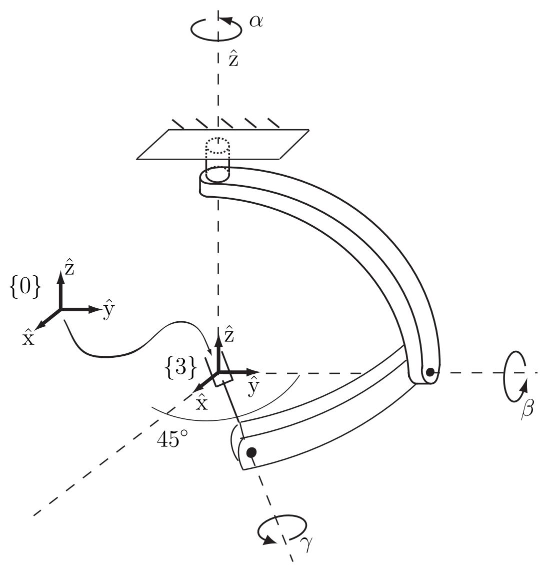


Figure 3.31: A three degree of freedom wrist mechanism.

(i) $R_{03} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(ii) $R_{03} = e^{[\hat{\omega}]\frac{\pi}{2}}$, where $\hat{\omega} = (0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}})$.

40. Refer to Appendix B.
(a) Verify the formula for obtaining the unit quaternion representation of a rotation $R \in SO(3)$.
(b) Verify the formula for obtaining the rotation matrix R given a unit quaternion $q \in S^3$.
(c) Verify the product rule for two unit quaternions. That is, given two unit quaternions $q, p \in S^3$ corresponding respectively to the rotations $R, Q \in SO(3)$, find a formula for the unit quaternion representation of the product $RQ \in SO(3)$.

41. (Refer to Appendix B.) The Cayley transform of Equation (B.18) can be generalized to higher-order as follows:

$$R = (I - [r])^k (I + [r])^{-k}. \tag{3.103}$$

(a) For the case $k = 2$, show that the rotation R corresponding to r can be computed from the formula

$$R = I + 4\frac{1 - r^T r}{(1 + r^T r)^2}[r] + \frac{8}{(1 + r^T r)^2}[r]^2. \tag{3.104}$$

(b) Conversely, given a rotation matrix R, show that a vector r that satisfies equation (3.104) can be obtained as

$$r = \hat{\omega} \tan \frac{\theta}{4}, \tag{3.105}$$

where as before $\hat{\omega}$ is the unit vector corresponding to the axis of rotation for R, and θ is the corresponding rotation angle. Is this solution unique?
(c) Show that the angular velocity in the body-fixed frame obeys the following relation:

$$\dot{r} = \frac{1}{4}\left\{(1 - r^T r)I + 2[r] + 2rr^T\right\}\omega. \tag{3.106}$$

(d) Explain what happens to the singularity at π that exists for the standard Cayley-Rodrigues parameters. Discuss the relative advantages and disadvantages of the modified Cayley-Rodrigues parameters, particularly as one goes to order $k = 4$ and higher.
(e) Compare the number of arithmetic operations for multiplying two rotation matrices, two unit quaternions, and two Cayley-Rodrigues representations. Which requires fewer arithmetic operations?

42. Among the programming languages for which the software exists, choose your favorite language. If it is not represented, port the Chapter 3 software to your favorite programming language.

43. Write a function that returns "true" if a given 3×3 matrix is within ϵ of being a rotation matrix, and "false" otherwise. It is up to you how to define the "distance" between a random 3×3 real matrix and the closest member of $SO(3)$. If the function returns "true," it should also return the "nearest" matrix in $SO(3)$. See, for example, Exercise 10.

44. Write a function that returns "true" if a given 4×4 matrix is within ϵ of an element of $SE(3)$, and "false" otherwise.

45. Write a function that returns "true" if a given 3×3 matrix is within ϵ of an element of $so(3)$, and "false" otherwise.

46. Write a function that returns "true" if a given 4×4 matrix is within ϵ of an element of $se(3)$, and "false" otherwise.

47. The primary purpose of the provided software is to be easy to read and educational, reinforcing the concepts in the book. The code is optimized neither for efficiency nor robustness, nor does it do full error-checking on its inputs.

Familiarize yourself with all of the code in your favorite language by reading the functions and their comments. This should help cement your understanding of the material in this chapter. Then:
(a) Rewrite one function to do full error-checking on its input, and have the function return a recognizable error value if the function is called with improper input (e.g., an argument to the function is not an element of $SO(3)$, $SE(3)$, $so(3)$, or $se(3)$, as expected).
(b) Rewrite one function to improve computational efficiency, perhaps by using what you know about properties of rotation or transformation matrices.
(c) Can you reduce the numerical sensitivity of either of the matrix logarithm functions?

48. Use the provided software to write a program that allows the user to specify an initial configuration of a rigid body by T, a screw axis specified by $\{q, \hat{s}, h\}$, and a total distance traveled along the screw axis θ. The program should calculate the final configuration $T_1 = e^{[\mathcal{S}]\theta}T$ attained when the rigid body follows the screw $\mathcal{S}$ a distance θ, as well as the intermediate configurations at $\theta/4$, $\theta/2$, and $3\theta/4$. At the initial, intermediate, and final configurations, the program should plot the {b} axes of the rigid body. The program should also calculate the screw axis $\mathcal{S}_1$, and the distance θ_1 following $\mathcal{S}_1$, that takes the rigid body from T_1 to the origin, and plot the screw axis $\mathcal{S}_1$. Test the program with $q = (0, 2, 0)^T$, $\hat{s} = (0, 0, 1)^T$, $h = 2$, $\theta = \pi$, and

$$T = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$