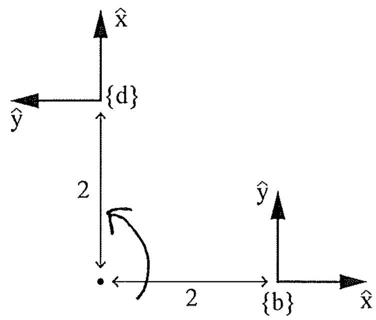
1. (4 pts) The end-effector frame {b} is shown below for a two-joint robot with a current joint vector $\theta^0 = [0, 2]^{\mathrm{T}}$, i.e., $T_{sb} = T(\theta^0)$. You would like to find a joint vector θ^* satisfying $T(\theta^*) = T_{sd}$, where {d} is also shown in the figure. You decide to use Newton-Raphson inverse kinematics, starting from θ^0, to approximately find θ^*. Assuming

$$J_b^{\dagger}(\theta^0) = \begin{bmatrix} 0 & 0 & 1/5 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

what is θ^1, your joint vector guess after one Newton-Raphson iteration?



$$\theta^1 = \theta^0 + J_b^{+} \mathcal{V}_b$$

$$[\mathcal{V}_b] = \log(T_{sb}^{-1} T_{sd})$$

$\mathcal{V}_b = \mathcal{S}_b \theta$. By inspection, $\mathcal{S}_b = (0,0,1,0,2,0)^T$ and $\theta = \pi/2$

$$\theta^1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1/5 & 0 & 2/5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \pi/2 \\ 0 \\ \pi \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} \pi/10 + 2\pi/5 \\ 0 \end{bmatrix} = \begin{bmatrix} \pi/2 \\ 2 \end{bmatrix}$$

2. (4 pts) The Lagrangian for a two-joint robot is $\mathcal{L} = \mathcal{L}^1 + \mathcal{L}^2 + \cdots$. One of the terms in the Lagrangian is $\mathcal{L}^1 = \mathfrak{m}_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2$. The dynamics of the robot can be expressed in the form

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{21}(\theta) & M_{22}(\theta) \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c_1(\theta, \dot{\theta}) \\ c_2(\theta, \dot{\theta}) \end{bmatrix} + \begin{bmatrix} g_1(\theta) \\ g_2(\theta) \end{bmatrix}.$$

Use the Euler-Lagrange equations to find the contributions of $\mathcal{L}^1$ to τ_1 and τ_2. For each term in your solution, indicate whether it would contribute to an $M_{11}, M_{12}, M_{21}, M_{22}, c_1, c_2, g_1$, or g_2 term.

$$\tau_1 = \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{\theta}_1} - \cancel{\frac{\partial \mathcal{L}'}{\partial \theta_1}}^{\,0}$$

$$= \frac{d}{dt}\left(m_2 \dot{\theta}_2 \sin\theta_2\right)$$

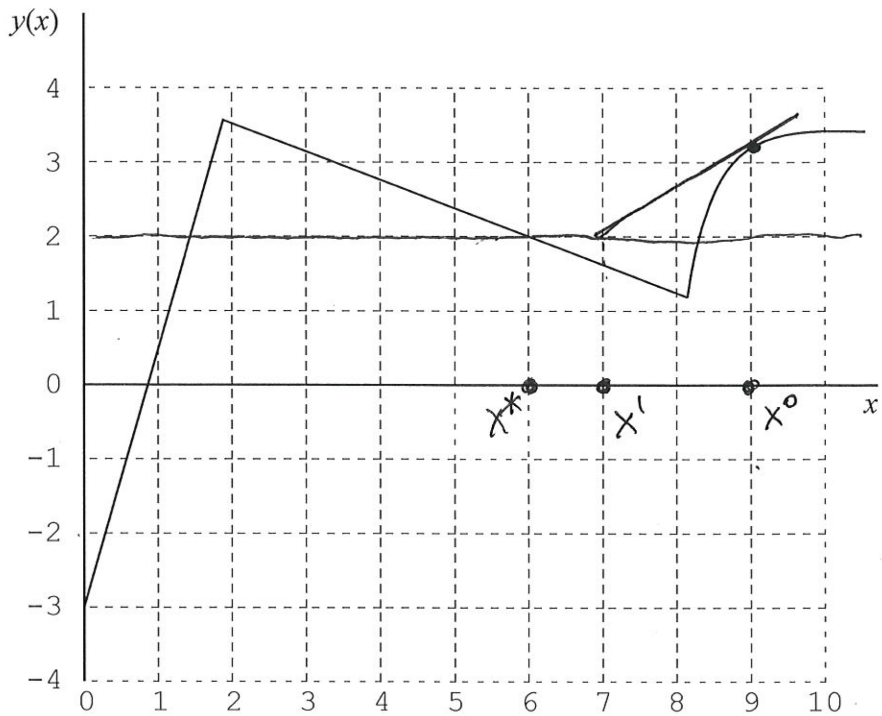
$$= \underbrace{m_2 \sin\theta_2 \ddot{\theta}_2}_{M_{12}} + \underbrace{m_2 \cos\theta_2 \dot{\theta}_2^2}_{c_1}$$

$$\tau_2 = \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{\theta}_2} - \frac{\partial \mathcal{L}'}{\partial \theta_2}$$

$$= \underbrace{\frac{d}{dt}\left(m_2 \dot{\theta}_1 \sin\theta_2\right)}_{m_2 \sin\theta_2 \ddot{\theta}_1 + m_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_2} - m_2 \dot{\theta}_1 \dot{\theta}_2 \cos\theta_2$$

$$= \underbrace{m_2 \sin\theta_2 \ddot{\theta}_1}_{M_{21}}$$

3. Consider the function $y(x)$ shown below. You will use iterative Newton-Raphson to try to find a value x^* satisfying $y(x^*) = 2$. Your Newton-Raphson guesses will always be limited to the range 0 to 10.



(a) (2 pts) Your initial guess at a solution is $x^0 = 9$. On the graph above, approximately but clearly draw one iteration of Newton-Raphson to calculate your next guess, x^1. What is the approximate value of x^1, as indicated by your drawing?

$x' \approx 7$

(b) (2 pts) What solution x^* will the Newton-Raphson process converge to (approximately)? Explain.

$x^* = 6$.

$y(x') \approx 1.6$. $\frac{\partial y}{\partial x}(x')$ is a slope exactly aligned with the actual function (a line between $x \approx 2$ and $x \approx 8$). The next iteration converges exactly.

(c) (2 pts) The stopping condition for the iterative process is $|2 - y(x)| < 0.1$. How many total iterations do you expect the process to take to meet the stopping condition? Explain.

2. We have exact convergence after 2 iterations, $|2 - y(x^2)| = 0$, because the function is a line between $x \approx 2$ and $x \approx 8$.

4. (2 pts) Two massive bodies, a and b, are rigidly attached to each other. The spatial inertias of each body are $\mathcal{G}_a$ and $\mathcal{G}_b$, respectively, measured in frames {a} and {b} at the bodies' centers of mass. The configuration of {b} in {a} is written T_{ab}. The rotational and translational dynamics of the composite rigid body can be written

$$(*) \quad \mathcal{F}_a = Z\dot{\mathcal{V}}_a - [\mathrm{ad}_{\mathcal{V}_a}]^{\mathrm{T}} Z\mathcal{V}_a.$$

Give Z in terms of $\mathcal{G}_a$, $\mathcal{G}_b$, and T_{ab}.

(*) is the dynamics of a rigid body expressed in {a}, so Z is the total spatial inertia expressed in {a}.

$$Z = \mathcal{G}_a + [\mathrm{Ad}_{T_{ab}^{-1}}]^{T} \mathcal{G}_b [\mathrm{Ad}_{T_{ab}^{-1}}]$$

5. The dynamics of a mass-spring-damper system are $4\ddot{x}+8\dot{x}+12x = f$, where $f = K_p x_e + K_d \dot{x}_e$ is the PD control law you will design. The error is $x_e = x_d - x$, and $x_d(t)$ is constant.

(a) (4 pts) For $x_d(t) = 0$, you would like the controlled system to have critical damping and, starting from $x_e(0) = 1$, the error should settle to values in the range $|x_e| < 0.02$ in approximately 1 second. What K_p and K_d do you choose?

$$x_e = x_d - x = -x$$
$$\dot{x}_e = -\dot{x}$$
$$\ddot{x}_e = -\ddot{x}$$

$$4\ddot{x} + 8\dot{x} + 12x = K_p x_e + K_d \dot{x}_e$$

$$4\ddot{x}_e + (8+K_d)\dot{x}_e + (12+K_p)x_e = 0$$

$$\ddot{x}_e + \underbrace{\frac{8+K_d}{4}}_{2\zeta\omega_n,\ \zeta=1,\ \text{so } 2\omega_n}\dot{x}_e + \underbrace{\frac{12+K_p}{4}}_{\omega_n^2} x_e = 0$$

$$t_s(2\%) = \frac{4}{\zeta\omega_n} = \frac{4}{\omega_n} \Rightarrow \omega_n = 4 = \sqrt{\frac{12+K_p}{4}} \Rightarrow K_p = 52$$

$$2\omega_n = 8 = \frac{8+K_d}{4} \Rightarrow K_d = 24$$

(b) (2 pts) For $x_d(t) = c$ (a constant nonzero position), what is the steady-state error $x_e(t \to \infty)$ for $K_p = K_d = 10$?

$$x_e = c - x$$
$$\dot{x}_e = -\dot{x}$$
$$\ddot{x}_e = -\ddot{x}$$

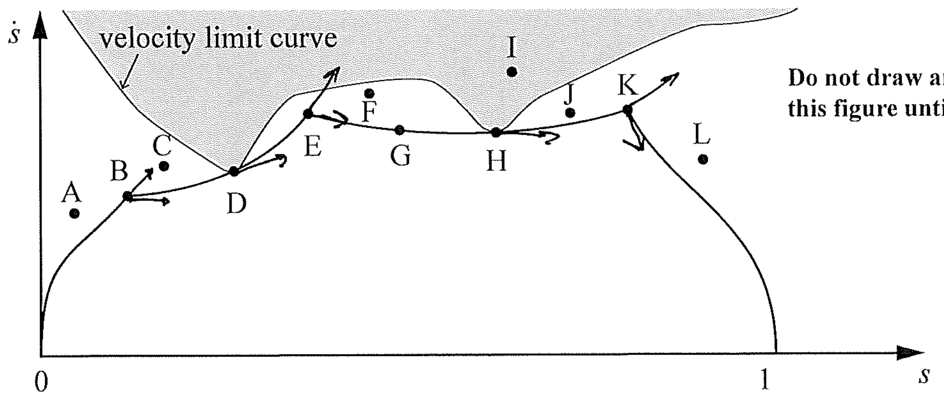
$$4\ddot{x} + 8\dot{x} + 12x = 10x_e + 10\dot{x}_e$$

$$-4\ddot{x}_e - 8\dot{x}_e - 12x_e + 12c = 10x_e + 10\dot{x}_e$$

At steady state, $\dot{x}_e = \ddot{x}_e = 0$

$$22x_e = 12c, \quad \text{so} \quad x_e = \frac{6}{11}c$$

6. The figure below shows a time-optimal time scaling, a velocity limit curve, and example $(s, \dot{s})$ states labeled as points A–L. **Do not draw anything on this figure until part (f).**



Do not draw anything on this figure until part (f).

(a) (2 pts) Which labeled states on the time-optimal time scaling correspond to a switch from maximum deceleration to maximum acceleration? (List all of them.)

$\boxed{D, H}$

(b) (2 pts) Which labeled states certainly cannot be reached from the initial state $(s, \dot{s}) = (0, 0)$? (List all of them.) Explain.

$\boxed{A, I, J}$

A: above the max accel curve

J: cannot be reached from H by max accel

I: interior of the "no go" region; can't get there

(c) (2 pts) From which labeled states will the robot certainly be unable to stay on the path ~~$\theta(s), s \in$~~ $[0, 1]$, all the way to the end state $(s, \dot{s}) = (1, 0)$? (List all of them.) Explain.

$\boxed{C, F, I, L}$

I: leaves path immediately

C, F: max decel hits velocity limit curve

L: cannot stop until $s > 1$

(d) (2 pts) This robot has two actuators. If we pick a random state on the time-optimal trajectory, how many actuators are likely to be saturated (operating at full capacity) at that state, 0, 1, or 2? Explain.

1 actuator. Should be more than zero for max acc/dec, and highly unlikely to be 2 simultaneously.

(e) (2 pts) At a state $(s, \dot{s})$, $L_1(s, \dot{s}) = -1, L_2(s, \dot{s}) = 3, U_1(s, \dot{s}) = 1$, and $U_2(s, \dot{s}) = 4$. Which of the labeled points could be at this $(s, \dot{s})$? (List all of them.) Explain.

I. For this, $L = \max(L_1, L_2) > U = \min(U_1, U_2)$

so no possibility to stay on the path.

(f) (2 pts) Draw the motion cones (tangent vector cones) at each of the labeled states where it is possible to know the angles of the cone vectors. (Don't worry about the lengths of the cone vectors; focus on the angles.) Don't draw anything at other points.

B, D, E, H, K