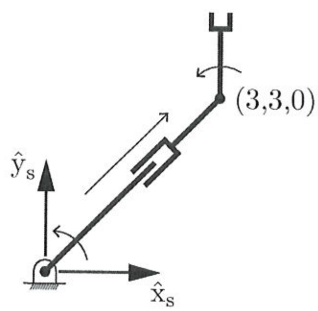
ME 449 Quiz 2
November 13, 2019

Name: Solutions

Where appropriate, show your work or reasoning so your thought process is clear. If you need more space for your work, you can use the back sides of the pages. No electronics (phone, tablet, computer, calculator, etc.). It is fine to have numerical answers that include ratios (e.g., 5/3), products (e.g., 47×14), or other standard functions (e.g., $\sqrt{3}$).

1. **(6 points)** A planar RPR robot, shown below, has a frame {s} at joint 1. The robot moves in the plane $z_s = 0$. Joint 3 is at the point $(3, 3, 0)$ in {s}. The direction of positive motion at each joint is shown. The end-effector twist satisfies $\mathcal{V}_s = J_s(\theta)\dot{\theta}$. $J_s(\theta)$ is a 6×3 matrix, since the twist is six-dimensional and there are only three joints.



Write the 6×3 space Jacobian $J_s(\theta)$ at the configuration shown. All entries should be numbers.

By inspection of the figure:

$$\begin{array}{c}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1/\sqrt{2} & 3 \\ 0 & 1/\sqrt{2} & -3 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \mathcal{S}_1 & \mathcal{S}_2 & \mathcal{S}_3 \end{matrix}\end{array}$$

if config shown is home config

2. **(2 points)** The mass matrix of a 2-dof robot at a particular configuration θ is

$$M(\theta) = \begin{bmatrix} a & -2 \\ -2 & 1 \end{bmatrix}.$$

The eigenvalues of M are $\lambda_{1,2} = (1/2)(1 + a \pm \sqrt{a^2 - 2a + 17})$. What constraint(s) must a satisfy for M to be a valid mass matrix?

Since M is symmetric, all e-vals are real. Since it's positive definite, all e-vals are positive. a must be real.

real: $a^2 - 2a + 17 \geq 0$

this is satisfied for all $a > 4$

positive: $1 + a > \sqrt{a^2 - 2a + 17}$

$(1+a)^2 > a^2 - 2a + 17$

$a^2 + 2a + 1 > a^2 - 2a + 17$

$4a > 16 \rightarrow \boxed{a > 4}$

3. **(8 points)** Consider two different robots with velocity kinematics $\dot{x} = J(\theta)\dot{\theta}$, where $\dot{x} \in \mathbb{R}^2$. (We only care about the 2-dof position x of the end effector, and $\dot{x}$ is its time derivative, so $J(\theta)$ is neither a space Jacobian nor a body Jacobian.) Robot 1 has two joints ($J(\theta) \in \mathbb{R}^{2\times 2}$) and robot 2 has three joints ($J(\theta) \in \mathbb{R}^{2\times 3}$). Each robot has configurations where the rank of its Jacobian matrix is two.

Below are Jacobian matrices at specific configurations of these robots. Beneath each Jacobian matrix, indicate whether the robot is redundant (which does not depend on the robot's configuration) and whether its configuration is singular. Your answer for each of the four matrices should be one of the following five choices: (1) "singular," (2) "redundant," (3) "singular and redundant," (4) "neither" (if neither singular nor redundant), or (5) "don't know" if you don't have enough information to completely answer the question.

robot 1 at θ_1	robot 1 at θ_2	robot 2 at θ_3	robot 2 at θ_4
$J = \begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix}$	$J = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$	$J = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$J = \begin{bmatrix} 1 & 2 & 10 \\ 2 & 4 & 20 \end{bmatrix}$
(1) singular since $\begin{bmatrix} -6 \\ -4 \end{bmatrix} = -2\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	(4) neither	(2) redundant	(3) singular + redundant $\begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \end{bmatrix} = 10\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

4. **(4 points)** The dynamic equations for joint i of a robot are $\tau_i = \text{term1} + \text{term2} + \ldots$. For each term below, label it using a number 1 to 5, where (1) means it could be a mass matrix term in the equation for τ_i, (2) means it could be a centripetal term in τ_i, (3) means it could be a Coriolis term in τ_i, (4) means it could be a potential term in τ_i, and (5) means that such a term could never appear in the equation for τ_i. The entire joint configuration is θ; θ_i refers to the configuration of joint i; and $f(\theta)$ is some function of θ (and possibly gravity or masses, inertias, or lengths of robot links).

(a) $f(\theta)$ (4) potential

(b) $f(\theta)\ddot{\theta}_3$ (1) mass matrix term

(c) $f(\theta)\ddot{\theta}_2\dot{\theta}_3$ (5) can't have product of joint velocity & acceleration

(d) $f(\theta)\dot{\theta}_2^2$ (2) centripetal

5. Consider two rigid bodies, 1 and 2. The mass of body 1 is 5 kg and the mass of body 2 is 10 kg. The spatial inertia of body 1 is $\mathcal{G}_1$ in a frame {1} at its center of mass, and the spatial inertia of body 2 is $\mathcal{G}_2$ in a frame {2} at its center of mass. The origin of frame {1} is at (0, 0, 0) in a fixed frame {s}, and the origin of frame {2} is at (0, 0, 10) in {s}.

(a) **(2 points)** If the two bodies are rigidly attached to each other, forming a single compound rigid body B with mass 15 kg, where is the center of mass of B in the {s} frame? Your answer should be a numerical 3-vector.

$$CM = \frac{5(0,0,0) + 10(0,0,10)}{15} = \frac{m_1 CM_1 + m_2 CM_2}{m_1 + m_2} = \boxed{\left(0, 0, \frac{100}{15}\right)}$$

(b) **(2 points)** Let {b} be a frame at the center of mass of the compound rigid body B. We know T_{1b} and T_{2b} and their inverses T_{b1} and T_{b2}. What is the spatial inertia $\mathcal{G}_b$ of the compound rigid body in terms of these transformation matrices and $\mathcal{G}_1$ and $\mathcal{G}_2$?

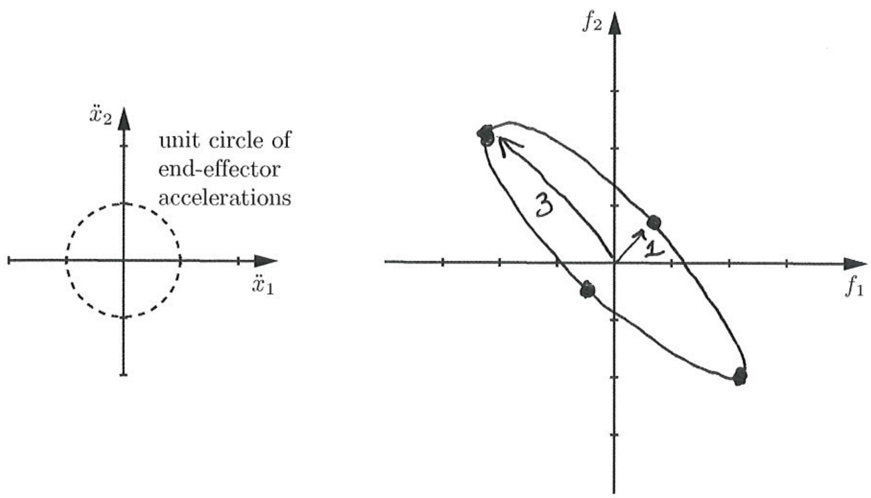
$$\boxed{\mathcal{G}_b = [Ad_{T_{1b}}]^T \mathcal{G}_1 [Ad_{T_{1b}}] + [Ad_{T_{2b}}]^T \mathcal{G}_2 [Ad_{T_{2b}}]}$$

from the fact $\frac{1}{2} \mathcal{V}_c^T \mathcal{G}_c \mathcal{V}_c = \frac{1}{2}\mathcal{V}_d^T \mathcal{G}_d \mathcal{V}_d = \frac{1}{2} \mathcal{V}_c^T \underbrace{[Ad_{T_{dc}}]^T \mathcal{G}_d [Ad_{T_{dc}}]}_{\mathcal{G}_c} \mathcal{V}_c$

6. **(6 points)** The position of the end-effector of a planar 2R robot is (x_1, x_2). The end-effector mass matrix is

$$\Lambda(\theta) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

with an eigenvalue of 3 in the direction $(-1, 1)$ and an eigenvalue of 1 in the direction $(1, 1)$. Use the axes on the right below to approximately draw the ellipse of end-effector forces (f_1, f_2) (the "end-effector mass ellipse") corresponding to a unit circle of end-effector accelerations (shown on the left) when the robot is at rest. Draw it to the proper scale, where each tick on the force axes corresponds to one unit of force. Mark the points on the ellipse where the direction of acceleration of the end-effector is the same as the direction of the applied force.



7. **(5 points)** Label each of the following statements "true" (T) or "false" (F) about the standard recursive Newton-Euler algorithm.

F (a) The algorithm calculates the robot's acceleration $\ddot{\theta}$ based on the current state $(\theta, \dot{\theta})$ and the joint torques τ. N-E is inverse dynamics

F (b) The algorithm first constructs a representation of the robot's mass matrix.

F (c) In the forward iterations, the calculation of the twist $\mathcal{V}_i$ of link i depends (in part) on $\dot{\theta}_i$, $\ddot{\theta}_i$, and $\mathcal{V}_{i-1}$. no dependence on $\ddot{\theta}_i$

F (d) The calculation of the acceleration $\dot{\mathcal{V}}_{i-1}$ of link $i-1$ depends (in part) on $\ddot{\theta}_i$. does not depend on accel of link after it

F (e) The calculation of τ_{i-1} depends (in part) on τ_i.

depends on $\mathcal{F}_i$, not τ_i