1. **(4 pts)** Consider three stationary frames, {a}, {b}, and {c}, and the transformation matrices $T_1 = T_{bc}$ and $T_2 = T_{ac}$. Give an expression for the twist $\mathcal{V}_c$ (expressed in the {c} frame) that moves a frame coincident with {a} to {b} in time t. Your answer should be in terms of T_1, T_2, t, and any appropriate operations on them. You may use the operator vec(), which takes an element of $se(3)$ and returns the associated six-vector representation. In other words, $\text{vec}([\mathcal{V}]) = \mathcal{V}$, i.e., vec() is the inverse of the [ ] operation.

$$T_{ab} = T_{ac} T_{cb} = T_2 T_1^{-1}$$

$\frac{1}{t} \log(T_2 T_1^{-1})$ is the se(3) rep of the twist that takes {a} to {b} in time t, represented in {a}. So,

$$\mathcal{V}_c = \frac{1}{t} [Ad_{T_{ca}}] \, \text{vec}(\log(T_2 T_1^{-1}))$$

$$\boxed{\mathcal{V}_c = \frac{1}{t} [Ad_{T_2^{-1}}] \, \text{vec}(\log(T_2 T_1^{-1}))}$$

2. **(3 pts)** A robot arm is at a configuration where the space Jacobian is J_s and the configuration of the end-effector frame {b} is given by T_{sb}. A wrench $\mathcal{F}_b$ is applied to the last link of the robot. What joint torques τ must the robot apply to remain stationary? Your answer should be in terms of J_s, T_{sb}, and $\mathcal{F}_b$ and any appropriate operations on them.

$\tau = -J_*^T \mathcal{F}_*$ where $*$ is s or b

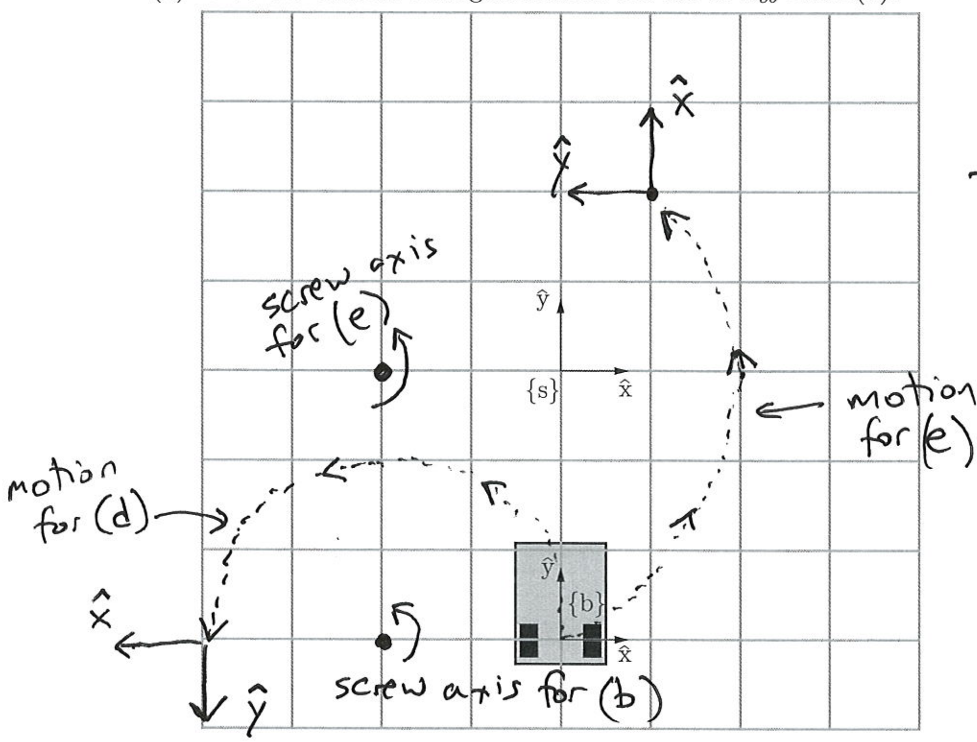
if $* = s$: $\tau = -J_s^T [Ad_{T_{bs}}]^T \mathcal{F}_b = \boxed{-J_s^T [Ad_{T_{sb}^{-1}}]^T \mathcal{F}_b}$

if $* = b$: $\tau = -([Ad_{T_{bs}}] J_s)^T \mathcal{F}_b$

$$= \boxed{\begin{aligned} &-([Ad_{T_{sb}^{-1}}] J_s)^T \mathcal{F}_b \\ =\ &-J_s^T [Ad_{T_{sb}^{-1}}]^T \mathcal{F}_b \end{aligned}}$$

3. **(10 pts)** In the figure below, the length of each square in the grid is 1, and there is a space frame {s} and a frame {b} between the rear wheels of the car.

(a) Give the current configuration of the car as $T_{sb} \in SE(3)$.



$$T_{sb} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) The car's chassis is moving with the twist $\mathcal{V}_b = (0, 0, 2, 0, 4, 0)^\intercal$. (i) Write the corresponding screw axis $\mathcal{B}$ expressed in {b} and (ii) in the figure in part (a), indicate the screw axis as a center of rotation (the point the chassis is instantaneously rotating about) along with the sense of rotation (clockwise or counterclockwise).

(i) screw axis must have $\|\omega\| = 1$, so $\mathcal{B} = \mathcal{V}_b / 2 = (0,0,1,0,2,0)^T$

(ii) screw axis must cause $\mathcal{V}_{by} = 2$. See figure.

(c) Assume the car follows the constant twist $\mathcal{V}_b$ for a time $t = \pi/2$, moving the car frame to {b′}. Give the exponential coordinate representation of $T_{bb'}$.

exp coords $= \mathcal{V}_b t = \mathcal{B}\theta = (0,0,2,0,4,0)\,\pi/2 = (0,0,\pi,0,2\pi,0)$

(d) After following the twist for time $t = \pi/2$, $T_{sb'} = T_{sb}\exp([\mathcal{V}_b]\pi/2)$. Give the numerical version of $T_{sb'}$. You should be able to visualize without evaluating equations. Explain your answer. You can use the figure.

Same as following $\mathcal{B}$ a distance $\theta = \pi$. See figure.

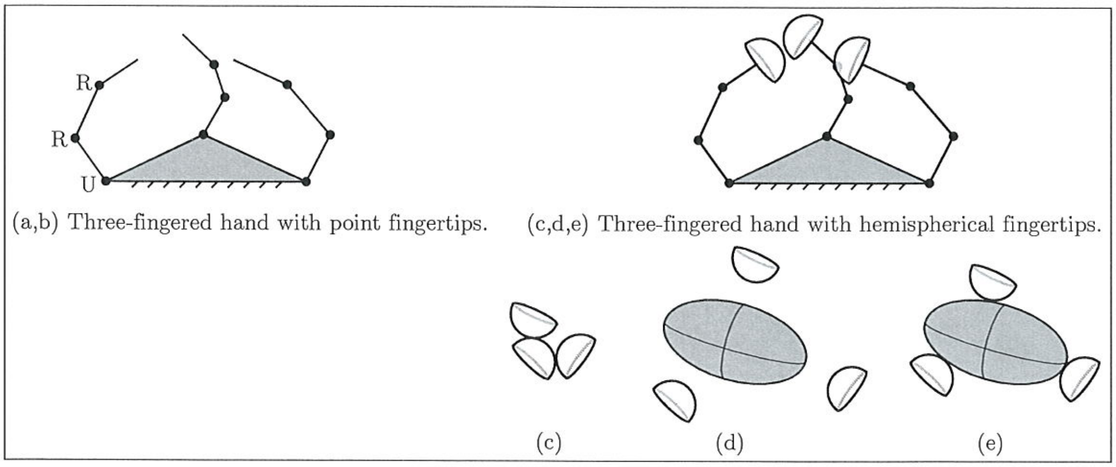
$$T_{sb'} = \begin{bmatrix} -1 & 0 & 0 & -4 \\ 0 & -1 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(e) Now give $\exp([\mathcal{B}]\pi/2)T_{sb}$. You should be able to visualize without evaluating equations. Explain your answer. You can use the figure.

Since the matrix exponential is on the left, the screw axis is expressed in {s}. See figure. Distance $\theta = \pi/2$.

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. **(10 pts)** The figure below shows a three-fingered robot hand with point fingertips (for questions a and b) and hemispherical fingertips (for questions c, d, and e). Each finger is a URR open chain. Each universal joint connects a finger to a stationary grounded base (palm).



(a,b) Three-fingered hand with point fingertips.

(c,d,e) Three-fingered hand with hemispherical fingertips.

(c) (d) (e)

(a) How many degrees of freedom does the 3×URR hand have, as shown in figure (a,b)?

Each finger has $2+1+1=4$ dof

$4\times 3 = 12$ dof

(b) If the point fingertips are all constrained to be at the same location (as if enforced by spherical joints), how many degrees of freedom does the system have? Explain.

First finger placed arbitrarily, last 2 have 3 constraints on their endpoint. Like 2 spherical joints. $12-(3\times 2) = 6$ dof

(c) Now the fingertips are hemispheres. How many degrees of freedom does the 3×URR hand have if each fingertip must touch at least one of the other fingertips, at any points of the hemispherical fingertips, as shown in figure (c)? Explain.

12 dof with no touching constraints. Each touching constraint reduces dof by 1 distance constraint between spheres. $12-2=10$ dof

(d) Now you introduce a rigid body to the system, as shown in figure (d), so the configuration of the complete system consists of the configuration of the rigid body and the configuration of the 3×URR hand (the fingertips are not constrained to touch the body). How many degrees of freedom does the complete system have? Explain.

12 dof for hand + 6 dof for body = 18 dof

(e) Now each fingertip of the 3×URR hand must touch the body, at any point on the body and fingertip, as shown in figure (e). How many degrees of freedom does the system have while satisfying this condition? Explain.

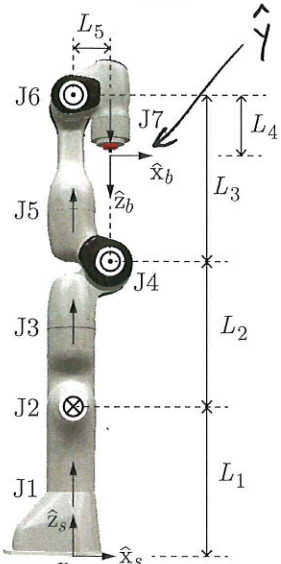
1 distance constraint between the body and each finger:

$18-(3\times 1) = 15$ dof

5. **(10 pts)** The 7R Franka Research FR3 robot arm is popular in research applications. As shown at its home configuration below, joint axes 1, 3, and 5 are up on the page, joint axis 7 is down, joint axis 2 is into the page, and joint axes 4 and 6 are out of the page. The figure also shows the space and end-effector frames, {s} and {b}, and all length offsets needed to fully specify the kinematics.

(a) Write $M = T_{sb}(0)$.

$\hat{y}$ out of page



$\hat{y}$ into page

$$M = \begin{bmatrix} 1 & 0 & 0 & L_5 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & L_1+L_2+L_3-L_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) Write the body Jacobian $J_b(0)$.

$$J_b(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & L_2+L_3-L_4 & 0 & -(L_3-L_4) & 0 & L_4 & 0 \\ -L_5 & 0 & -L_5 & 0 & -L_5 & 0 & 0 \\ 0 & L_5 & 0 & 0 & 0 & -L_5 & 0 \end{bmatrix}$$

columns 1, 3, 5 are identical

(c) At this home configuration, give a joint velocity $\dot{\theta}$ yielding $\mathcal{V}_b = (1, 0, 0, 0, 0, 0)^\intercal$.

All zeros in the top row of the Jacobian, so it is not possible to get this twist.

$$\mathcal{V}_b = J_b \dot{\theta}$$