**M2794.002700 Introduction to Robotics**
**Midterm Examination 1**
**April 12, 2018**
**CLOSED BOOK, CLOSED NOTES**

**Problem 1** (50 points)
(a) Determine the degrees of freedom for the three mechanisms shown below. Try to use an appropriate version of Grübler's formula to justify each of your answers; in cases where this is not possible, carefully explain your answer based on physical reasoning and intuition.

1. The $3 \times$ PRPS mechanism of Figure 1(a).
2. The $3 \times$ RRRRRSR mechanism of Figure 1(b).
3. The golfer of Figure 1(c): Assume that both feet are always firmly planted to the ground, and that the two "hands" are rigidly attached to the golf club.

(b) Figure 1(d) shows a rigid ball inside a spherical bowl. The ball is always in contact with the bowl. Assuming there is no slip between the bowl and the ball, what is the configuration space of the ball? What if the contact between the ball and the bowl is frictionless?

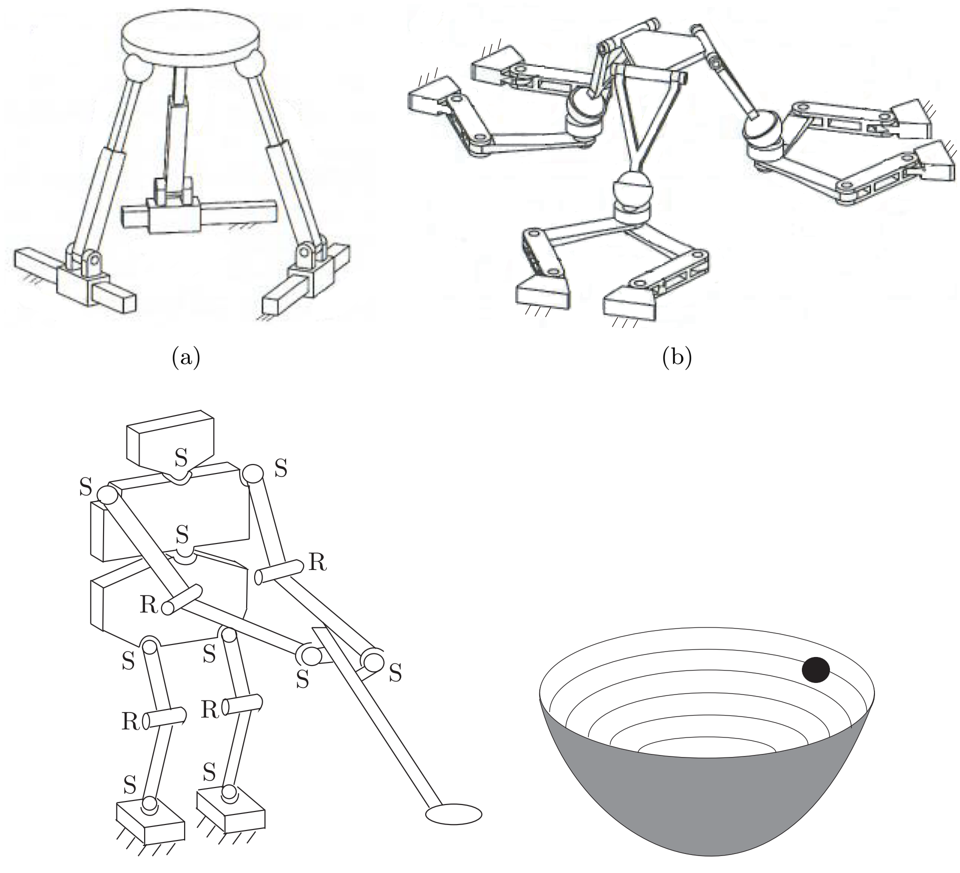
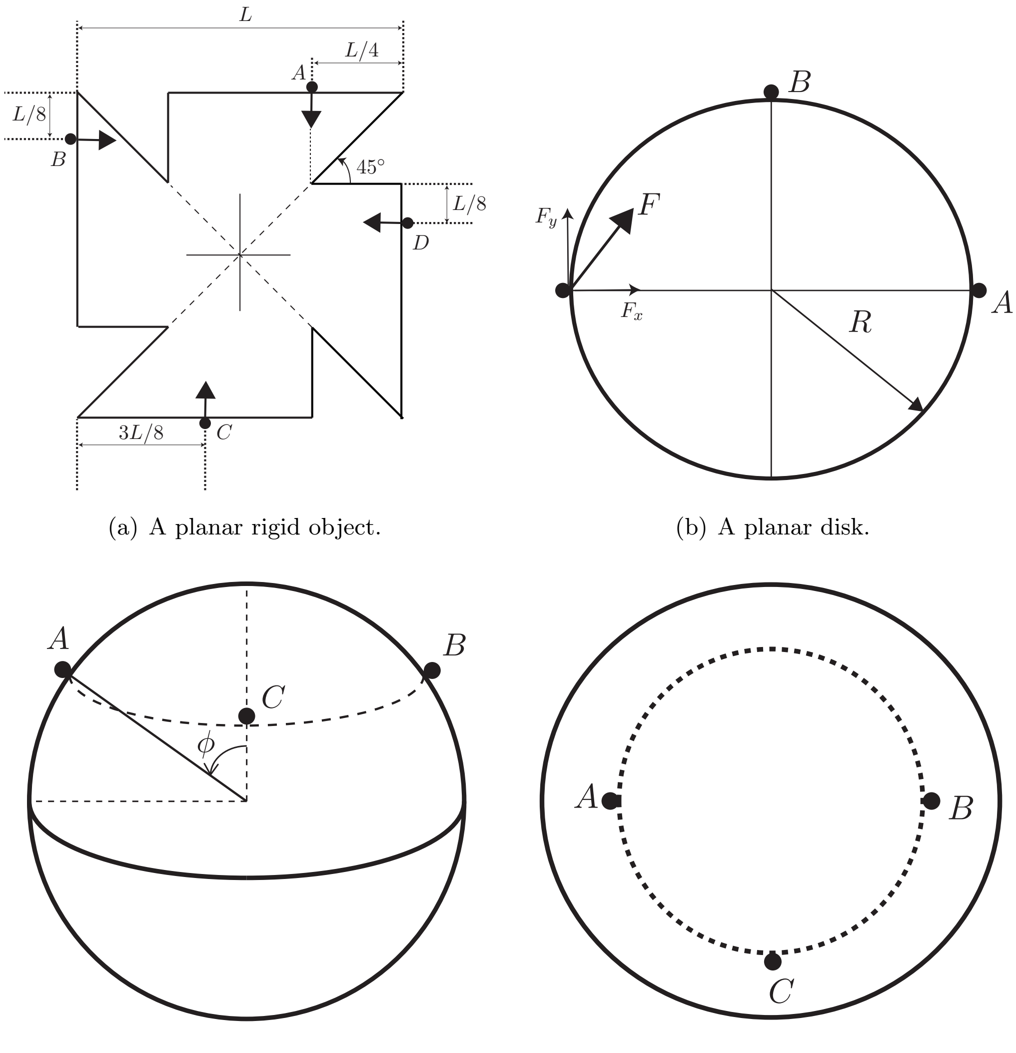



Figure 1: Mechanisms for Problem 1.

**Problem 2** (50 points)
(a) The planar rigid object of Figure 2(a) is grasped by four frictionless point contacts $A, B, C, D$ as shown. Is this grasp force closure?
(b) Now suppose point contact $D$ can be moved to anywhere on the object. Draw all possible locations for $D$ such that the grasp is force closure. You must explain your answer to receive full credit.
(c) A planar disk of radius $R$=1 is grasped by two point contacts $A, B$ as shown in Figure 2(b). Point contact $A$ has friction coefficient $\mu = 1$, while point contact $B$ is frictionless. Suppose an arbitrary external force $F = (F_x, F_y)$ is applied to the disk at the point shown in the figure. Find the complete range of all forces $F$ that can be resisted by the two point contacts $A, B$. Try to express your answer as an inequality involving $F_x$ and $F_y$.
(d) A three-dimensional rigid sphere of radius $R$=1 is grasped by three point contacts $A, B, C$ as shown in Figure 2(c). Assume that all contact points have the same friction coefficient $\mu = 1$. Determine the range of the angle $\phi$ so that the grasp remains force closure.




(a) A planar rigid object.

(b) A planar disk.

(c) A three-dimensional rigid sphere.

(d) Top view of the sphere.

Figure 2: Object grasps for Problem 2.

**Problem 3** (50 points)

(a) Find $R \in SO(3)$ corresponding to the ZXZ roll-pitch-roll angles $\alpha = 90°$, $\beta = 180°$, $\gamma = 45°$ (that is, given a moving frame originally at the identity configuration, rotate this frame about the fixed frame $Z$-axis by angle $\alpha$, then rotate about the fixed frame $X$-axis by angle $\beta$, then rotate about the fixed frame $Z$-axis by angle $\gamma$, and find the $R \in SO(3)$ corresponding to this final configuration).
(b) Find $R \in SO(3)$ corresponding to the ZXZ Euler angles $\alpha = 90°$, $\beta = 180°$, $\gamma = 45°$ (that is, given a moving frame originally at the identity configuration, rotate this frame about its $Z$-axis by angle $\alpha$, then rotate about its displaced $X$-axis by angle $\beta$, then rotate about its displaced $Z$-axis by angle $\gamma$, and find the $R \in SO(3)$ corresponding to this final configuration).
(c) Suppose an object is at initial orientation

$$R_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

We wish to rotate this object about some rotation axis $\hat{\omega} \in \mathbb{R}^3$, $\|\hat{\omega}\| = 1$, by some angle $\theta \in [0, \pi]$, to the new orientation

$$R_1 = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find $\hat{\omega}$ and $\theta$ that achieves this displacement.
(d) Two space stations, the Tiangong-1 (TG-1) and the International Space Station (ISS), are orbiting the earth as shown in Figure 3. Take the radius of the earth to be $R$, and assume an inertial frame $\{0\}$ is attached to the earth center, with angle $\theta$ measured counterclockwise about the inertial frame $Z$-axis as shown. The TG-1 rotates in a longitudinal circular orbit (i.e., in the $Y$-$Z$ plane of the inertial frame) with constant speed $v_1$ (refer to the left figure). The ISS rotates in another circular orbit, parallel to the $X$-$Y$ plane, of radius $R_2 = \sqrt{R_1^2 - \frac{R^2}{2}}$ and at a distance $R/\sqrt{2}$ from the earth center (refer to the right figure). The locations of the two frames at time $t = 0$ are as indicated in the figures. Moving reference frames $\{1\}$ and $\{2\}$ are respectively attached to the TG-1 and the ISS in the following manner:

- The $z$-axis of frame $\{1\}$ always points toward the earth center, while the $x$-axis always points in the direction of the TG-1's velocity vector;
- The $y$-axis of frame $\{2\}$ always points toward the center of the ISS orbit, while the $x$-axis points in the direction of the ISS's velocity vector.

Denote by $T_{ij} \in SE(3)$ the $4 \times 4$ rigid body transformation matrix describing the position and orientation of frame $\{j\}$ as seen from frame $\{i\}$. Find $T_{01}$ and $T_{02}$ as a function of time $t$.
(e) The TG-1 is expected to crash to earth on April 1, 2018, and by an inexplicable twist of fate, Matt Damon is aboard the TG-1 and must again be rescued. A rescue spacecraft is launched into a longitudinal circular orbit at $\theta = 0°$, moving at the same speed as the TG-1. The plan is for Matt Damon to extricate himself from the TG-1, and to use his jet pack to maneuver himself to rendezvous point P, where he will be rescued by the spacecraft. Assume frame $\{3\}$ is attached to the rescue spacecraft such that its $z$-axis always points in the outward radial direction and its $x$-axis points in the direction of its velocity vector. The position and orientation of the rescue spacecraft at time $t = 0$ is as shown in Figure 4. Derive $T_{03}$ as a function of $t$.
(f) Matt Damon's position can only be measured from the ISS; his coordinates with respect to

frame $\{2\}$ are $p_2 = \left(0, R_2, R_1 - \frac{R}{\sqrt{2}}\right)$. Find $p_3 \in \mathbb{R}^3$, his coordinates with respect to frame $\{3\}$, and also the time $T$ it will take for the rescue spacecraft to reach Matt Damon.
(g) After rescuing Matt Damon, the spacecraft must now rendezvous with the ISS at time $t = 3T/2$ to drop Matt off. The velocity vector of the ISS at $t = 3T/2$, expressed in the inertial frame $\{0\}$ coordinates, is given by $(0, -v_2, 0)$. What is the corresponding ISS velocity vector expressed in frame $\{3\}$ coordinates?

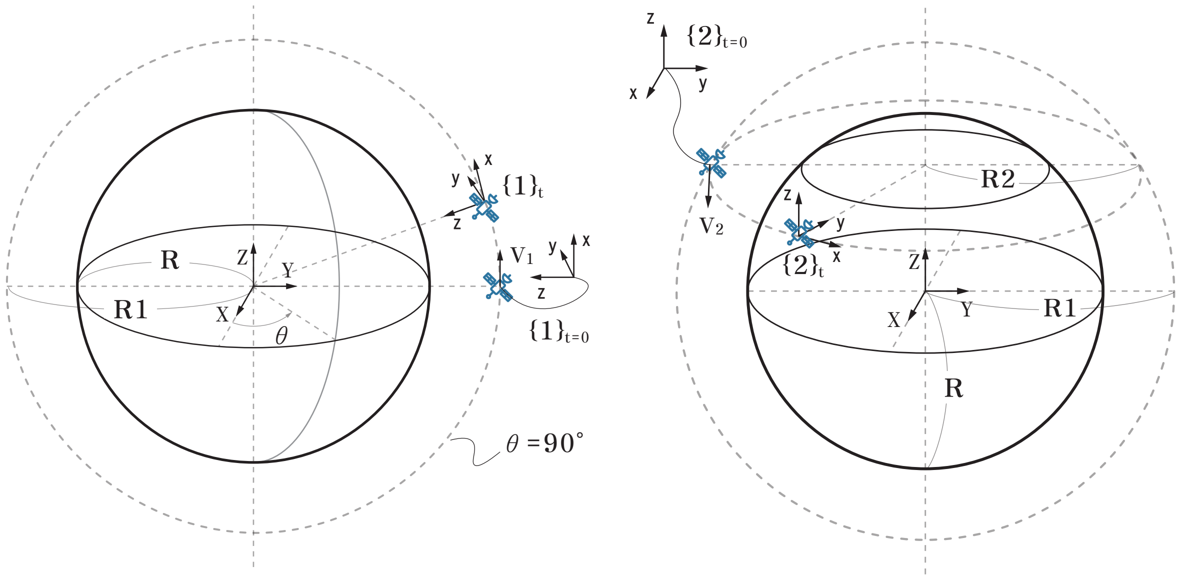



Figure 3: Problem 3(a)

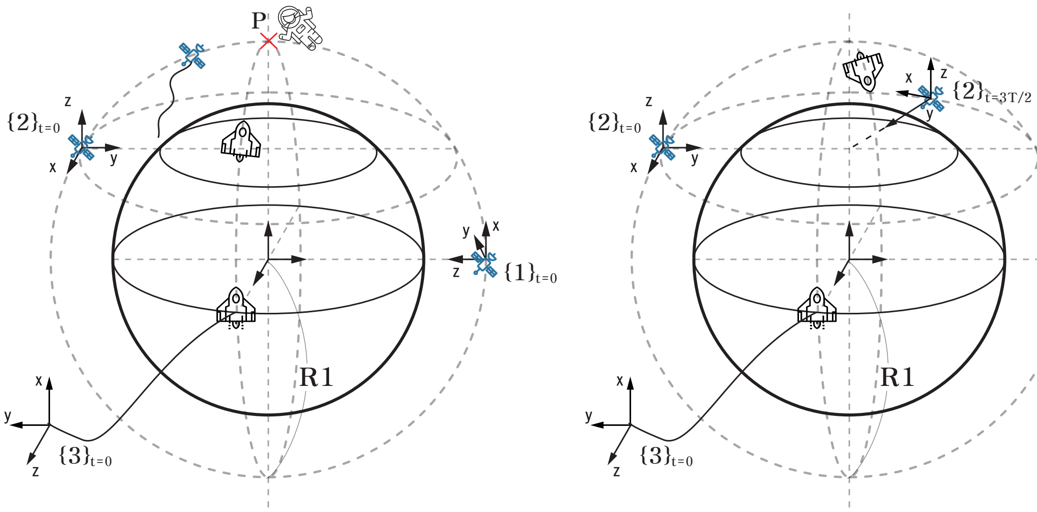
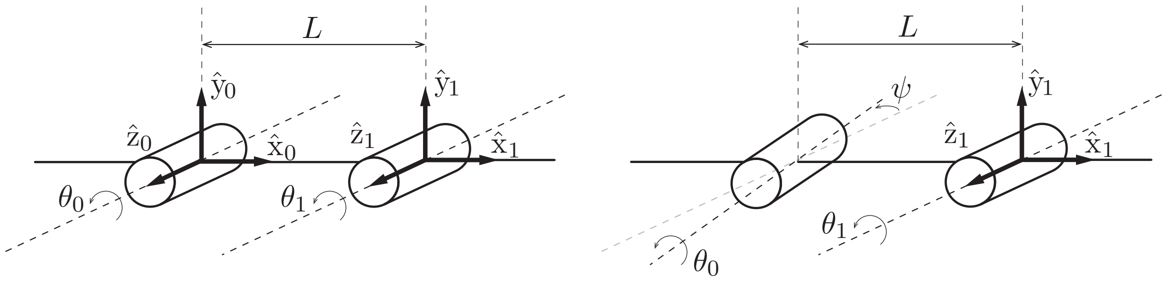



Figure 4: Figures for Problem 3(e)-(g)

**Problem 4** (50 points)
(a) Two parallel revolute joints of a robot are connected by a link as shown in Figure 5(a). For the link frames $\{0\}$ and $\{1\}$ assigned as shown and assuming the robot is in its zero position, find the corresponding Denavit-Hartenberg parameters for $T_{01}$.
(b) Due to imprecisions in the manufacturing process, joint axis 0 is not quite parallel to joint axis 1; instead, it is rotated by a small angle $\psi$ about the $y_0$-axis as shown in Figure 5(b). Assign a new link frame $\{0\}$ and find the corresponding Denavit-Hartenberg parameter $d_1$. Draw a graph of $d_1$ versus $\psi$ over the range $\psi \in [-\frac{\pi}{100}, \frac{\pi}{100}]$.

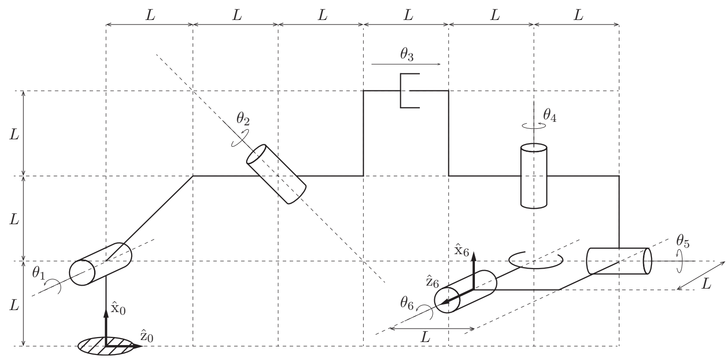



(a) Two parallel revolute joints for Problem 4(a).

(b) Two slightly misaligned revolute joints for Problem 4(b).

Figure 5: Two revolute joints for Problem 4(a) and 4(b).

(c) Now consider the RRPRRR spatial open chain of Figure 6, shown in its zero position. Frames $\{0\}$, $\{6\}$, and some Denavit-Hartenberg parameter values are given as shown in Figure 6(a) and Table 6(b). Draw appropriate link frames and fill in the rest of Table 6(b) with the corresponding Denavit-Hartenberg parameter values.
(d) For the same RRPRRR open chain shown in its zero position, denote by $M_{15} \in SE(3)$ the displacement of frame $\{5\}$ with respect to frame $\{1\}$. Find the end-effector pose $T_{06} \in SE(3)$ when $\theta_1 = \pi, \theta_2 = \theta_3 = \theta_4 = \theta_5 = 0, \theta_6 = \frac{\pi}{2}$. You may express your answer in terms of Rot() and Trans() operations.




(a)

| $i$ | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\phi_i$ |
|---|---|---|---|---|
| 1 | | | | |
| 2 | | | | |
| 3 | $\frac{5}{4}\pi$ | | | |
| 4 | $\frac{1}{2}\pi$ | | | |
| 5 | $\frac{3}{2}\pi$ | | | |
| 6 | | | | $\theta_6$ |

(b)

Figure 6: RRPRRR open chain for Problem 4(c)-(d).

**M2794.0027 Introduction to Robotics**
**Midterm Examination 1 Solutions**
**April 12, 2018**

**Problem 1**

(a) For each of the mechanisms we'll first try to apply an appropriate version of Grübler's formula, and see if the result agrees with our physical intuition:

1. Applying the spatial version of Grübler's formula leads to the following:
   $N = 7$ (links) $+1$ (ground) $= 8$
   $J = 3$ (S joints) $+$ 3 (P joints) $+$ 3 (PR joints) $= 9$
   $\Sigma f_i = 3 \times 3$ (S joints) $+1 \times 3$ (P joints) $+2 \times 3$ (PR joints) $= 18$
   $\mathrm{dof} = 6(N - 1 - J) + \Sigma f_i = 6(8 - 1 - 9) + 18 = 6$
   Observe that if the three prismatic joints and three revolute joints at the base are locked, then unless if the legs are all parallel, it is impossible for the legs to extend in length, implying that the mechanism becomes a structure in this case. Therefore this mechanism has six dof.

2. The first thing to note is that this mechanism is a hybrid planar-spatial mechanism: the spatial portion of the mechanism is a $3 \times$ RS platform, while the three planar RRRR five-bar linkages essentially act as two-dof XY positioning devices for the base locations of the three legs. Unfortunately the figure did not make clear how the spherical joints are attached to the planar five-bar linkages; below we describe solutions for two possible interpretations of this connection. In the first case, if the socket of each spherical joint is assumed rigidly attached to one of the coupler links (i.e., one of the two middle links of the five-bar), then naively applying the spatial version of Grübler's formula leads to
   $N = 16$ (links) $+1$ (ground) $= 17$
   $J = 3$ (S joints) $+$ 18 (R joints) $= 21$
   $\Sigma f_i = 3 \times 3$ (S joints) $+1 \times 18$ (R joints) $= 27$
   $\mathrm{dof} = 6(N - 1 - J) + \Sigma f_i = 6(17 - 1 - 21) + 27 = -3$
   Grübler's formula would thus seem to imply that the mechanism is overconstrained. However, each planar five-bar linkage can be regarded as a two-dof XY Cartesian positioning device, or XY joint. Applying the spatial version of the Grübler's formula to this interpretation yields
   $N = 7$ (links) $+1$ (ground) $= 8$
   $J = 3$ (S joints) $+$ 3 (R joints) $+$ 3 (XY joints) $= 9$
   $\Sigma f_i = 3 \times 3$ (S joints) $+1 \times 3$ (R joints) $+2 \times 3$ (XY joints)$= 18$
   $\mathrm{dof} = 6(N - 1 - J) + \Sigma f_i = 6(8 - 1 - 9) + 18 = 6$

   A second interpretation of the spherical joint connection is that the socket is attached to the five-bar linkage in a way such that the socket can also rotate about its fixed axis (torsional rotation). In this case the socket lies above two overlapping R joints, with the socket regarded as link connected to one of these R joints. Applying the spatial version of Grübler's formula with this interpretation leads to
   $N = 10$ (links) $+1$ (ground) $= 11$
   $J = 3$ (S joints) $+$ 6 (R joints) $+$ 3 (XY joints) $= 12$
   $\Sigma f_i = 3 \times 3$ (S joints) $+1 \times 6$ (R joints) $+2 \times 3$ (XY joints)$= 21$
   $\mathrm{dof} = 6(N - 1 - J) + \Sigma f_i = 6(11 - 1 - 12) + 21 = 9$
   Since the torsional rotation about each socket has no effect on the motion of moving platform, it is customary to regard the degrees of freedom of this mechanism as six $(9 - 3)$.

3. Applying the spatial version of Grübler's formula leads to the following:
   $N = 12$ (links) $+1$ (ground) $= 13$
   $J = 10$ (S joints) $+$ 4 (R joints) $= 14$
   $\Sigma f_i = 3 \times 10$ (S joints) $+1 \times 4$ (R joints) $= 34$
   dof $= 6(N-1-J) + \Sigma f_i = 6(13-1-14) + 34 = 22$
   To determine if this agrees with our intuition, observe that the lower closed chain consisting of two legs and the lower torso has, from a straightforward application of the spatial version of Grübler's formula, eight degrees of freedom ($N = 6$, $J = 6$ (four S-joints and two R-joints), $\sum f_i = 14$). The closed chain formed by the upper body (assume for the moment that the upper body is stationary) and two arms grasping the golf club also has eight degrees of freedom. The neck and waist, both modeled as S-joints, each have three degrees of freedom, leading to a total of 8+8+3+3=22 degrees of freedom, consistent with our earlier calculation.

(b) Assuming no friction between the ball and spherical bowl, the ball can then assume any position and orientation inside the bowl (while maintaining contact with the bowl as stipulated by the problem). The configuration space of the ball is then $S^2 \times SO(3)$, where $S^2$ denotes the two-dimensional sphere (or more precisely in this case, the bowl is represented by a half-sphere). If on the other hand the contact between the ball and bowl was no-slip, then one can legitimately ask whether the ball can in fact reach any arbitrary configuration in $S^2 \times SO(3)$ purely from rolling motions. The answer is yes: showing this is far from trivial, but to illustrate how this is possible with an example from everyday life, imagine how a car can, through a combination of steering and moving forward and backward, can reach arbitrary positions and orientations. (The ball rolling without slip is an example of a **nonholonomic** system, a subject addressed in the textbook but which we have not yet covered in this class.)

**Problem 2**

(a) Express the static equilibrium force closure conditions in the standard linear form $Ax = b$, where $A \in \mathbb{R}^{3\times 4}$ is given, and the objective is to determine whether a nonnegative solution $x \geq 0$ exists for any arbitrary $b \in \mathbb{R}^3$. Setting up the problem in this way leads to the Gauss-Jordan elimination of the following matrix:

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ -L/4 & -3L/8 & -L/8 & L/8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 2/3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2/3 \end{bmatrix}.$$

Force closure requires that all entries of the fourth column be negative; since this is not the case, the grasp is not force closure. (**Note**: Another popular method of solution (apparently covered in the discussion sessions but not in lecture) was to effectively transform the problem into a form that is amenable to Nguyen's Theorem: the point contacts at $B$ and $C$ are merged into a single frictional point contact with friction cone whose edges are collinear with the normal contact forces at $B$ and $C$ (see Figure 1); point contacts $A$ and $D$ can be similarly merged as illustrated. Now, viewing this grasp as a two-finger point contact grasp with friction at the two internal points of the grasped object as indicated, Nguyen's Theorem implies that this grasp cannot be force closure, since the line connecting the two point contacts (indicated in red) lies outside both friction cones. While technically this analysis is correct, invoking Nguyen's theorem in this manner does require further justification of its correctness.)

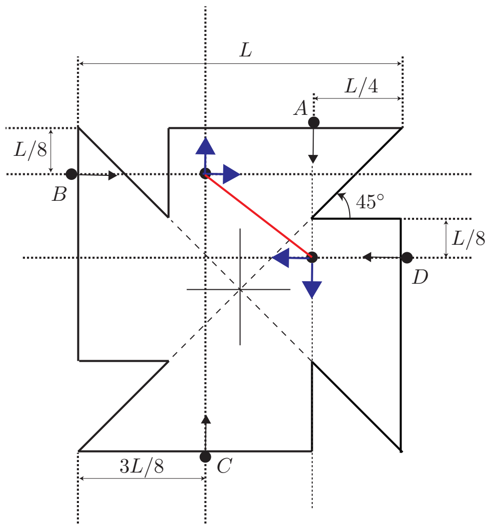
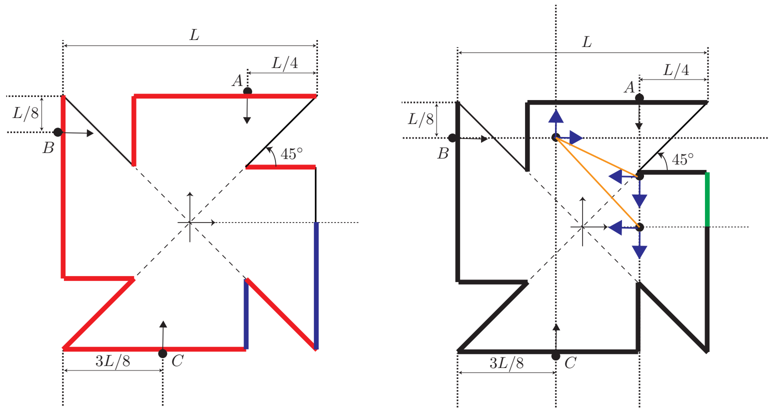



Figure 1: Planar rigid object for Problem 2(a).

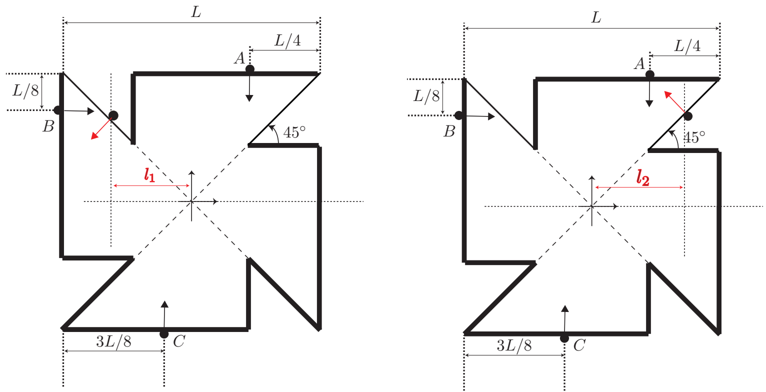
(b) We'll solve this problem in a number of steps:

- **Step 1**: In order to resist a positive x-directional force, point contact D cannot be placed on the red line in Figure 2(a). Similarly, in order to resist a negative moment, the contact cannot be placed on the blue line of the same figure.



(a) Planar rigid object (b) Planar rigid object

Figure 2: Solution to Problem 2(b)



(a) The planar rigid object (b) The planar rigid object

Figure 3: Solution to Problem 2(b).

- **Step 2**: Now specify the contact locations among the three remaining regions. For contacts placed on the green line in Figure 2(b), the grasp cannot be force closure by the same reasoning given in (a).
- **Step 3**: Use Gauss-Jordan elimination for the two remaining blade-like edges:
    1. Determine the range of $l_1$ in Figure 3(a):

    $$\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & -1 \\ -L/4 & -3L/8 & -L/8 & 2l_1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 + \frac{4L-16l_1}{3L} \\ 0 & 0 & 1 & \frac{4L-16l_1}{3L}. \end{bmatrix}$$

    If $\frac{4L-16l_1}{3L} < 0$, then the grasp is fore closure. Thus, $l_1 > \frac{L}{4}$ must be satisfied to be force closure.
    2. Determine the range of $l_2$ in Figure 3(b):

    $$\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \\ -L/4 & -3L/8 & -L/8 & 2l_2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 + \frac{2L-16l_2}{3L} \\ 0 & 0 & 1 & \frac{2L-16l_2}{3L}. \end{bmatrix}$$

If $1 + \frac{2L - 16l_2}{3L} < 0$, then the grasp is fore closure. Thus, $l_2 > \frac{5L}{16}$ must be satisfied for the grasp to be force closure.

All possible locations for contact D are indicated by the two purple lines in Figure 4.

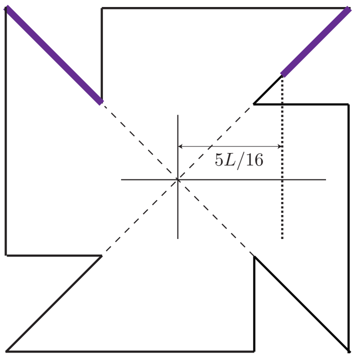
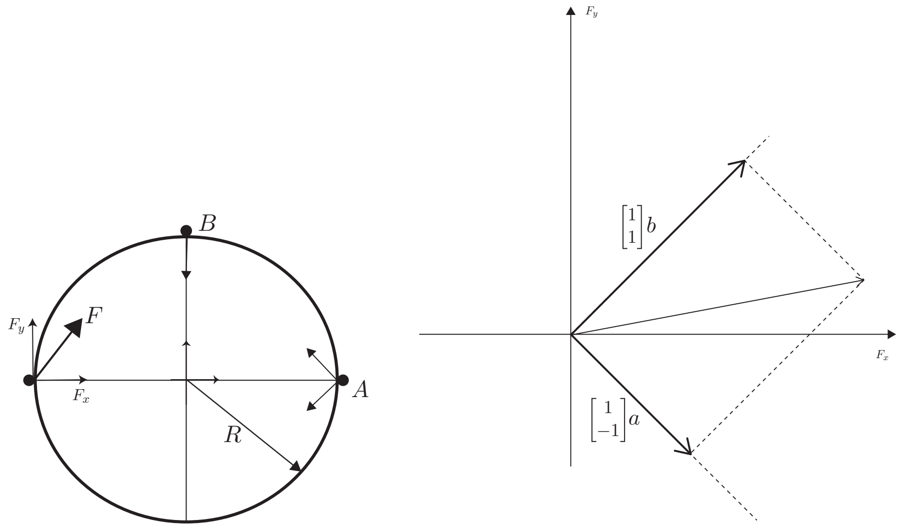



Figure 4: Planar rigid object for Problem 2(b).




(a) The planar disk

(b) The vector subspace

Figure 5: Solution to Problem 2(c)

(c) Suppose an arbitrary force $F$ is applied to the point as shown in Fig 5-(a). Assume that the object is in static equilibrium, then express the force and moment equilibrium equations in the usual matrix form $Ax = b$ and perform Gauss-Jordan elimination:

$$\begin{vmatrix} -1 & -1 & 0 \\ -1 & 1 & -1 \\ -R & R & 0 \end{vmatrix} \begin{vmatrix} -F_x \\ -F_y \\ R \times F_y \end{vmatrix} \Longrightarrow \begin{vmatrix} 1 & 1 & 0 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{vmatrix} \begin{vmatrix} F_x \\ 0 \\ F_y \end{vmatrix}.$$

The forces that can be resisted by the two point contacts $A, B$ are therefore characterized by the

$a, b, c \geq 0$ that satisfy

$$\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} a + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} b + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} c = \begin{bmatrix} F_x \\ 0 \\ F_y \end{bmatrix}.$$

From the middle equation we obtain $b - a = c/2 \geq 0$, or $b \geq a$. The first and third equations lead to

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} a + \begin{bmatrix} 1 \\ 1 \end{bmatrix} b = \begin{bmatrix} F_x \\ F_Y \end{bmatrix}.$$

The above conditions on $(F_x, F_y)$ simplify to $F_y \geq 0$ and $F_y \leq F_x$, or a positive cone in the first quadrant of the $F_x$-$F_y$ plane, bounded by the $F_x$-axis and the line $F_y = F_x$.
(d) Given a three-dimensional rigid sphere of radius $R = 1$ grasped by three point contacts A, B, C as shown in Figure 6, with uniform friction coefficient $\mu = 1$, the problem asks for the range of angle $\phi$ such that the grasp remains force closure. We recount two facts about spatial force closure:

- Given a spatial rigid body restrained by three point contacts with friction, assume that the three contact points lie on a unique plane $S$, and the friction cone at each of the contacts intersects $S$ in a cone. The body is in force closure if and only if the the plane $S$ is in a planar force closure grasp.

- If the intersected friction cone lies in the plane $S$, it always satisfies the conditions for Nguyen's Theorem in $S$ as shown in Figure 6.

For the given problem, since $\mu = 1$, $\alpha = 45°$. When $\phi > 45°$, the intersected cone lies in $S$. From symmetry considerations the requirement for the grasp to be force closure is $45° < \phi < 135°$.

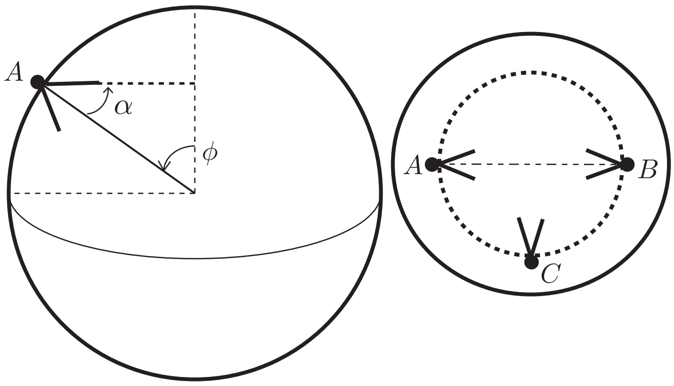



Figure 6: Three-dimensional sphere for Problem 2(d).

**Problem 3**

(a) The rotation $R \in SO(3)$ corresponding to the given ZXZ roll-pitch-yaw angles $\alpha = 90°$, $\beta = 180°$, $\gamma = 45°$ is given by $R = \text{Rot}(\hat{z}, 45°)\text{Rot}(\hat{x}, 180°)\text{Rot}(\hat{z}, 90°)$, or

$$R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(b) The rotation $R \in SO(3)$ corresponding to the given ZXZ Euler angles $\alpha = 90°$, $\beta = 180°$, $\gamma = 45°$ is given by $R = \text{Rot}(\hat{z}, 90°)\text{Rot}(\hat{x}, 180°)\text{Rot}(\hat{z}, 45°)$, or

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(c) Find the unit vector $\hat{\omega}$ and angle $\theta \in [0, 2\pi]$ such that $R_1 = e^{[\hat{\omega}]\theta} R_0$. Denoting $e^{[\hat{\omega}]\theta}$ by $R$, we have

$$R = R_1 R_0^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The rotation angle $\theta$ must satisfy $1 + 2\cos\theta = \text{tr}(R)$, or for our problem, $\cos\theta = \frac{\sqrt{2}}{2} - \frac{1}{4}$:

$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2} - \frac{1}{4}\right). \tag{1}$$

Say $\alpha \in [0, \pi]$ is one solution to (1); then $2\pi - \alpha$ is another solution. The rotation axis $\hat{\omega}$ is given by

$$[\hat{\omega}] = \frac{1}{2\sin\theta}\left(R - R^T\right) = \frac{1}{2\sin\theta} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} - \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} + \frac{1}{2} & 0 & \frac{1}{\sqrt{2}} + \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{\sqrt{2}} - \frac{1}{2} & 0 \end{bmatrix}.$$

We thus have two solutions:

$$\theta = \alpha, \quad \hat{\omega} = \frac{2}{\sqrt{7 + 4\sqrt{2}}} \begin{bmatrix} -\frac{1}{\sqrt{2}} - \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} + \frac{1}{2} \end{bmatrix}$$

and

$$\theta = 2\pi - \alpha, \quad \hat{\omega} = -\frac{2}{\sqrt{7 + 4\sqrt{2}}} \begin{bmatrix} -\frac{1}{\sqrt{2}} - \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{\sqrt{2}} + \frac{1}{2} \end{bmatrix}$$

(d) For convenience let $\omega_1 = v_1/R_1$ and $\omega_2 = v_2/\sqrt{R_2^2 - \frac{R_2^2}{2}}$. Place the frame $\{a\}$ as shown in Figure 7; the origin is at the center of the earth and the orientation is $\text{Rot}(Y, -90°)\text{Rot}(X, 90°)$

with respect to the $\{0\}$ frame. We can also place the moving frame $\{a'\}$ such that the origin is at the center of the earth and the orientation is $\text{Rot}\,(y_a, -\omega_1 t)$. With these frame assignments we have

$$\begin{aligned} R_{0a} &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \\ R_{aa'} &= \begin{bmatrix} \cos\omega_1 t & 0 & -\sin\omega_1 t \\ 0 & 1 & 0 \\ \sin\omega_1 t & 0 & \cos\omega_1 t \end{bmatrix}. \end{aligned}$$

Therefore we have

$$R_{01} = R_{0a'} = R_{0a}R_{aa'} = \begin{bmatrix} 0 & -1 & 0 \\ -\sin\omega_1 t & 0 & -\cos\omega_1 t \\ \cos\omega_1 t & 0 & -\sin\omega_1 t \end{bmatrix}.$$

Because the TG-1 rotates in a longitudinal circular orbit in the Y-Z plane of the inertial frame with constant speed $v_1$,

$$p_{01} = \begin{bmatrix} 0 \\ R_1\cos\omega_1 t \\ R_1\sin\omega_1 t \end{bmatrix}.$$

We thus obtain the solution

$$T_{01} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\sin\omega_1 t & 0 & -\cos\omega_1 t & R_1\cos\omega_1 t \\ \cos\omega_1 t & 0 & -\sin\omega_1 t & R_1\sin\omega_1 t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Similarly, we can place frame $\{b\}$ as shown in Figure 7, with the origin at the center of the circle parallel to the $XY$ plane, and orientation $\text{Rot}\,(Z, \omega_2 t)$:

$$T_{0b} = \begin{bmatrix} \cos\omega_2 t & -\sin\omega_2 t & 0 & 0 \\ \sin\omega_2 t & \cos\omega_2 t & 0 & 0 \\ 0 & 0 & 1 & \frac{R}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{b2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -R_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Multiplying $T_{0b}$ and $T_{b2}$, we get

$$T_{02} = \begin{bmatrix} \cos\omega_2 t & -\sin\omega_2 t & 0 & R_2\sin\omega_2 t \\ \sin\omega_2 t & \cos\omega_2 t & 0 & -R_2\cos\omega_2 t \\ 0 & 0 & 1 & \frac{R}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(e) Let $\omega_3 = v_3/R_1$, so that $\omega_3 = \omega_1$. Let the frame $\{c\}$ be the initial frame of the spaceship. Since $R_{0c} = \text{Rot}\,(y, 90°)\,\text{Rot}\,(z, 180°)$ and $R_{c3} = \text{Rot}\,(y, \omega_3 t)$,

$$R_{0c} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

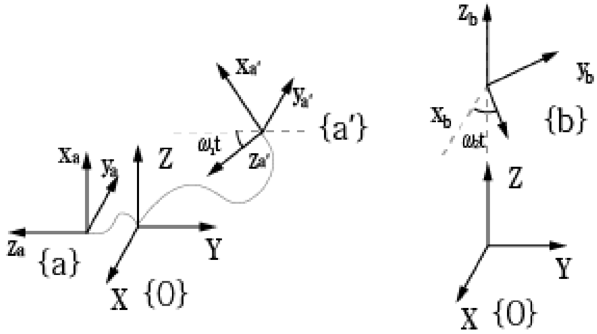



Figure 7: Solution to Problem 3(d)

and

$$R_{c3} = \begin{bmatrix} \cos\omega_3 t & 0 & \sin\omega_3 t \\ 0 & 1 & 0 \\ -\sin\omega_3 t & 0 & \cos\omega_3 t \end{bmatrix}.$$

Multiplying $R_{0c}$ and $R_{c3}$, we get

$$R_{03} = \begin{bmatrix} -\sin\omega_3 t & 0 & \cos\omega_3 t \\ 0 & -1 & 0 \\ \cos\omega_3 t & 0 & \sin\omega_3 t \end{bmatrix}.$$

Because the spacecraft rotates in a longitudinal circular orbit in the X-Z plane of the inertial frame with constant speed $v_1$,

$$p_{01} = \begin{bmatrix} R_1 \cos\omega_3 t \\ 0 \\ R_1 \sin\omega_3 t \end{bmatrix},$$

where $\omega_1 = \omega_3$. We therefore get

$$T_{03} = \begin{bmatrix} -\sin\omega_3 t & 0 & \cos\omega_3 t & R_1 \cos\omega_3 t \\ 0 & -1 & 0 & 0 \\ \cos\omega_3 t & 0 & \sin\omega_3 t & R_1 \sin\omega_3 t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(f) We present two possible solutions to this problem:
**Solution 1**: Let $p_i(t)$ be Matt Damon's coordinates with respect to frame $\{i\}$ at time $t$. $p_2(t)$ is constant, since Matt Damon is at the north pole P. From Figure 8, $p_3(t)$ can be calculated as

$$p_3(t) = \begin{bmatrix} R_1 \cos\omega_1 t \\ 0 \\ R_1(\sin\omega_1 t - 1) \end{bmatrix}.$$

Observe that $p_3(t)$ is zero at $t = \pi/2\omega_1$.

**Solution 2**: Using the $T_{03}$ obtained in problem 3(e),

$$T_{03} = \begin{bmatrix} -\sin\omega_3 t & 0 & \cos\omega_3 t & R_1 \cos\omega_3 t \\ 0 & -1 & 0 & 0 \\ \cos\omega_3 t & 0 & \sin\omega_3 t & R_1 \sin\omega_3 t \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

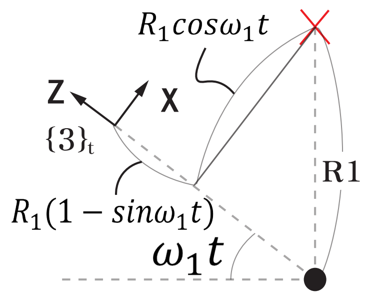



Figure 8: Solution to Problem 3(f)

$$T_{30} = T_{03}^{-1} = \begin{bmatrix} -\sin\omega_3 t & 0 & \cos\omega_3 t & 0 \\ 0 & -1 & 0 & 0 \\ \cos\omega_3 t & 0 & \sin\omega_3 t & -R_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since Matt Damon is at the north pole P, $p_0 = (0, 0, R_1)^T$. $p_3\,(t)$ is therefore

$$p_3\,(t) = T_{30} p_0 = \begin{bmatrix} -\sin\omega_3 t & 0 & \cos\omega_3 t & 0 \\ 0 & -1 & 0 & 0 \\ \cos\omega_3 t & 0 & \sin\omega_3 t & -R_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ R_1 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1 \cos\omega_1 t \\ 0 \\ R_1\,(\sin\omega_1 t - 1) \\ 1 \end{bmatrix}.$$

(g) The spacecraft and the ISS rendezvous at $t = \frac{3}{2}T$ (the orientation of the ISS and the spacecraft at $t = \frac{3}{2}T$ are shown in Figure 9.) Let $v_{ISS}\,(t)$ be the velocity vector of the ISS. Then $v_{ISS}\,(t)$ in frame {2} can be expressed as $(v_2, 0, 0)$, since the $x$-axis points in the direction of the ISS velocity vector. As shown in Figure 9, $v_{ISS}(3T/2)$ in frame {3} can be expressed as $(0, v_2, 0)$.

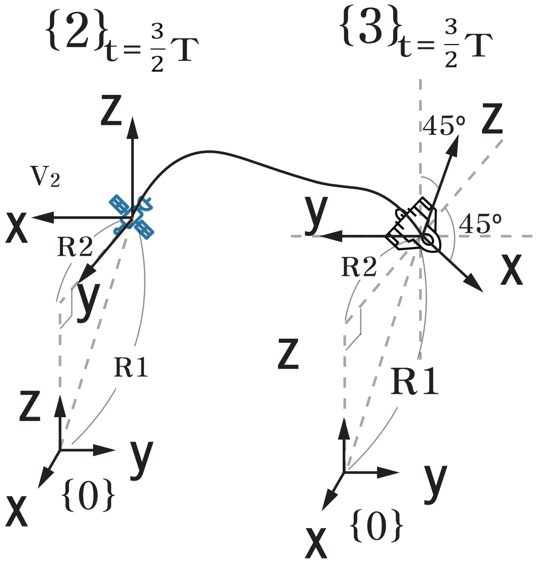
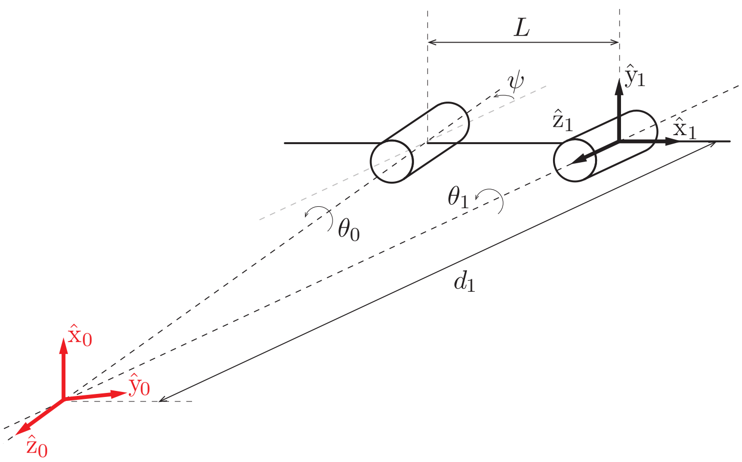



Figure 9: Solution to Problem 3(g)

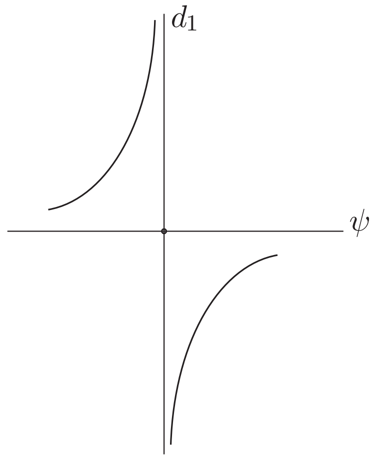
**Problem 4**

(a) $\alpha_0 = 0, a_0 = L, d_1 = 0, \phi_1 = \theta_1$.

(b) One possible new link frame $\{0\}$ is shown in Figure 10, while a second solution is to set $\hat{x}_0$ in the opposite direction. In both cases the parameter $d_1 = -L \cot \psi$ for $\psi \neq 0$, and $d_1 = 0$ for $\psi = 0$. Note that $d_1$ is discontinuous at $\psi = 0$.




(a) Attached link frame $\{0\}$.




(b) Graph of $d_1$ vs $\psi$.

Figure 10: Solution to Problem 4(b).

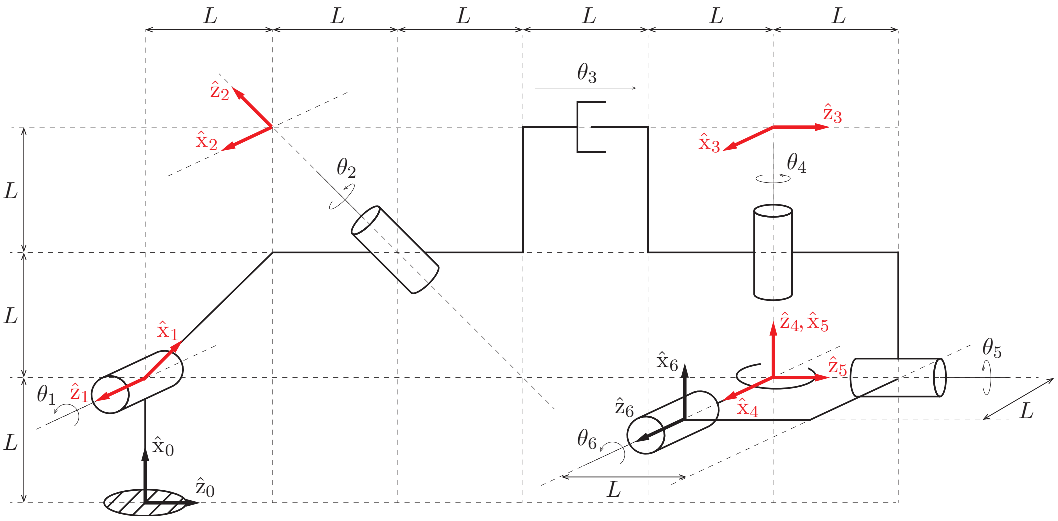
(c) An appropriate set of link frames are assigned as shown in Figure 11(a). The corresponding Denavit-Hartenberg parameter values are listed in the Table shown in 11(b).

(d) Using the Denavit-Hartenberg parameter values obtained in (c), we get

$$\begin{aligned}
T_{01} &= \mathrm{Rot}(x, \frac{3\pi}{2})\mathrm{Trans}(x, L)\mathrm{Rot}(z, \frac{3\pi}{4}) \\
T_{15} &= M_{15} \\
T_{56} &= \mathrm{Rot}(x, \frac{3\pi}{2})\mathrm{Trans}(z, L)\mathrm{Rot}(z, \frac{\pi}{2}).
\end{aligned}$$

from which it follows that

$$\begin{aligned}
T_{06} &= T_{01}T_{15}T_{56} \\
&= \mathrm{Rot}(x, \frac{3\pi}{2})\mathrm{Trans}(x, L)\mathrm{Rot}(z, \frac{3\pi}{4})M_{15}\mathrm{Rot}(x, \frac{3\pi}{2})\mathrm{Trans}(z, L)\mathrm{Rot}(z, \frac{\pi}{2}).
\end{aligned}$$




(a)

| $i$ | $\alpha_{i-1}$ | $a_{i-1}$ | $d_i$ | $\phi_i$ |
|---|---|---|---|---|
| 1 | $\frac{3}{2}\pi$ | $L$ | 0 | $\frac{7}{4}\pi + \theta_1$ |
| 2 | $\frac{3}{2}\pi$ | $\frac{3\sqrt{2}}{2}L$ | $\frac{\sqrt{2}}{2}L$ | $\frac{3}{2}\pi + \theta_2$ |
| 3 | $\frac{5}{4}\pi$ | 0 | $4L + \theta_3$ | 0 |
| 4 | $\frac{1}{2}\pi$ | 0 | -2$L$ | $\theta_4$ |
| 5 | $\frac{3}{2}\pi$ | 0 | 0 | $\frac{3}{2}\pi + \theta_5$ |
| 6 | $\frac{3}{2}\pi$ | 0 | $L$ | $\theta_6$ |

(b)

Figure 11: Attached link frames and their corresponding Denavit-Hartenberg parameters.

**M2794.002700 Introduction to Robotics**
**Midterm Examination 2**
**May 24, 2018**
**CLOSED BOOK, CLOSED NOTES**

**Problem 1** (50 points)
While the Dark Lord Thanos is admiring his just-acquired Infinity Gauntlet, Groot seizes this opportunity to snatch it away from Thanos's hand. Modeling Groot's left arm as a spatial RRPRPR open chain as shown in Figure 1 (shown in its zero position), with fixed frame {s}, end-effector frame {b}, and gauntlet frame {g}, answer the following questions.
(a) Express Groot's arm forward kinematics in the form

$$T_{sb} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_6]\theta_6} M,$$

where $M \in \mathrm{SE}(3)$ and $\mathcal{S}_1, \ldots, \mathcal{S}_6 \in \mathrm{se}(3)$.
(b) Assume Groot's left arm is in the configuration $\theta = (\frac{\pi}{3}, 0, 0, -\frac{2\pi}{3}, L, 0)$. Find the end-effector spatial velocity $\mathcal{V}_s$ when the input joint velocity vector is $\dot{\theta} = (1, 1, 1, 1, 1, 1)$. Also, express the linear velocity of the {b}-frame origin in {s}-frame coordinates.
(c) With $\theta_2$ fixed to $\theta_2 = 0$, Groot now attempts to grab the gauntlet, which is located at

$$T_{sg} = \begin{bmatrix} R_{sg} & p_{sg} \\ 0 & 1 \end{bmatrix}, \ R_{sg} = \mathrm{Rot}(\hat{x}, \alpha)\mathrm{Rot}(\hat{y}, \beta), \ p_{sg} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}.$$

Derive an analytic procedure for solving the inverse kinematics; that is, find all possible solutions $\theta_i$ in the range $-\pi < \theta_i \leq \pi$, $i = 1, 3, 4, 5, 6$, in terms of the given parameters $\alpha, \beta, p_x, p_y, p_z$ $(-\pi < \alpha, \beta \leq \pi)$. How many inverse kinematics solutions are there?

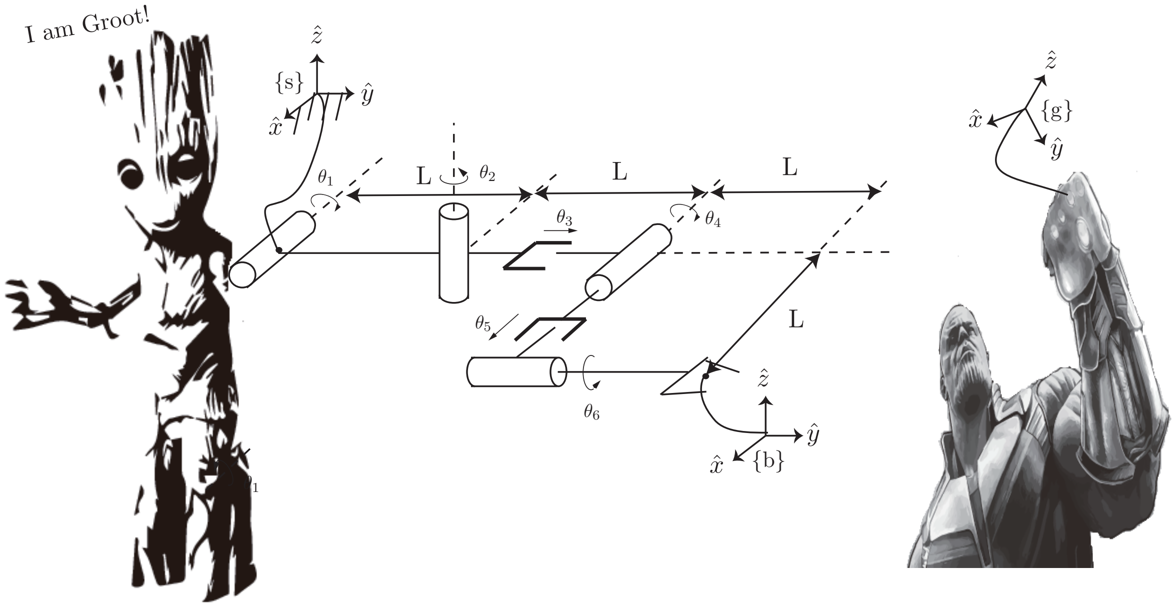



Figure 1: Groot's left arm shown in its zero position.

**Problem 2** (50 points)
Given a six-dof open chain manipulator with forward kinematics

$$T = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M e^{[\mathcal{B}_4]\theta_4} e^{[\mathcal{B}_5]\theta_5} e^{[\mathcal{B}_6]\theta_6},$$

where

$$\mathcal{S}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathcal{S}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{B}_4 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{B}_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathcal{B}_6 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and

$$M = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Sketch the manipulator as accurately as you can.
(b) Determine the last three columns of the body Jacobian $J_b(\theta)$.
(c) Determine if the zero position is a kinematic singularity.

**Problem 3** (50 points)
Figure 2 shows a planar manipulator with a cam attached to joint $\theta_1$. As $\theta_1$ rotates, the link in contact with the cam translates linearly along the $y$-axis. The shape of the cam is described in Figure 2, with $r(\theta) = 3R + R\cos\theta$.
(a) Derive the Jacobian relating the joint rates $(\dot{\theta}_1, \dot{\theta}_2)$ to the tip velocity $(\dot{x}, \dot{y})$.
(b) For your Jacobian derived in (a), find all kinematic singularities and the corresponding directions in which the tip loses mobility.
(c) Assume $R = L = 1$ and draw the corresponding manipulability ellipsoids for the following two configurations:

1. $\theta_1 = \frac{\pi}{2}$, $\theta_2 = \frac{\pi}{2}$;
2. $\theta_1 = 0$, $\theta_2 = \frac{\pi}{4}$.

(d) One scalar measure of a robot's manipulability is given by

$$\mu_1 = \frac{\sqrt{\lambda_{max}(JJ^T)}}{\sqrt{\lambda_{min}(JJ^T)}},$$

which is the ratio of the lengths of the longest and shortest principal axes of the manipulability ellipsoid; when $\mu_1$ is one, then the manipulability ellipsoid becomes a circle. For the given manipulator, find the ratio of $R$ and $L$ that minimizes $\mu_1$ at the configuration $\theta_1 = \frac{\pi}{4}$, $\theta_2 = \frac{\pi}{4}$.
(e) Another scalar measure of a robot's manipulability is the volume of the manipulability ellipsoid:

$$\mu_2 = \sqrt{\det(JJ^T)}.$$

A configuration with a larger value of $\mu_2$ is considered to have better manipulability. Assuming $L = R = 1$ for the given manipulator, find the configuration $(\theta_1, \theta_2)$ that maximizes $\mu_2$.

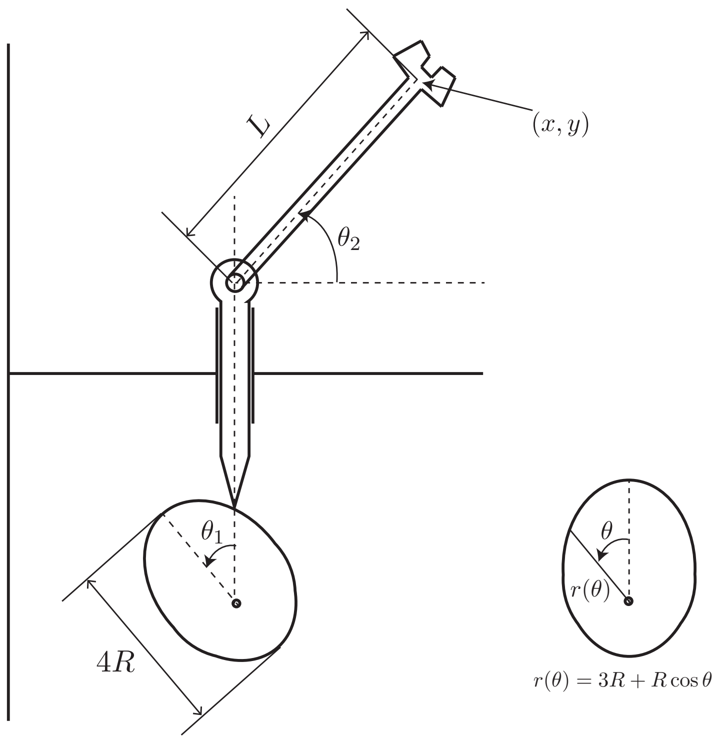



Figure 2: Planar manipulator for Problem 3.

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**Problem 4** (50 points)
(a) Recall that the transformation rules for spatial velocities and spatial forces are given respectively by the adjoint and its transpose, i.e.,

$$\mathcal{V}_b = [\mathrm{Ad}_{T_{ba}}]\mathcal{V}_a, \quad \mathcal{F}_b = [\mathrm{Ad}_{T_{ab}}]^T \mathcal{F}_a.$$

where $\mathcal{V}$ and $\mathcal{F}$ are of the form $\mathcal{V} = (\omega, v)$, $\mathcal{F} = (m, f)$. Instead of writing spatial forces as above, Roy likes to reverse the order of $m$ and $f$ as follows:

$$\mathcal{F}' = \left[ \begin{array}{c} f \\ m \end{array} \right].$$

By defining spatial forces in this way, Roy claims that the transformation rule for $\mathcal{F}'$ is given by $\mathcal{F}'_b = [\mathrm{Ad}_{T_{ba}}]\mathcal{F}'_a$. Is Roy correct?
(b) Let {b} be a reference frame attached to the center of mass of a rigid body, and let $\mathcal{I}_b$ be the $3\times3$ inertia matrix of this rigid body with respect to frame {b}. The mass of the body is $\mathfrak{m}$. Suppose another reference frame {a} is attached to another point on the rigid body, and $T_{ab} \in \mathrm{SE}(3)$ is known. What is the inertia matrix $\mathcal{I}_a \in \mathbb{R}^{3\times3}$ of the rigid body with respect to this new frame {a}?
(c) Given an $n$-link open chain, suppose reference frame $\{i\}$ is attached to the center of mass of link $i$, and

$$T_{0i} = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_i]\theta_i} M_i,$$

be the forward kinematics to frame $\{i\}$. Let $\mathcal{V}_i$ be the spatial velocity of frame $\{i\}$ expressed in the fixed frame {s}, i.e., $[\mathcal{V}_i] = \dot{T}_{0i} T_{0i}^{-1}$. Find a recursive formula for $\mathcal{V}_i$ in terms of $\mathcal{V}_{i-1}$, $\mathcal{S}_i$, $\dot{\theta}_i$, and any link frame transformations as needed.

**M2794.0027 Introduction to Robotics**
**Midterm Examination 2 Solutions**
**May 24, 2018**

## Problem 1

(a) The forward kinematics is expressed in the space form of the product-of-exponentials formula, where $M \in \mathrm{SE}(3)$ is the end-effector configuration at the zero position and $\mathcal{S}_1, \ldots, \mathcal{S}_6 \in \mathrm{se}(3)$ are the twists corresponding to the joint axes when the robot is at its zero position. As seen from the figure, $M \in SE(3)$ is given by

$$M = \begin{bmatrix} 1 & 0 & 0 & L \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Values of $\mathcal{S}_i = (\omega_i, v_i)$, $i = 1, \ldots, 6$ are listed below:

| Axis $i$ | $w_i$ | $q_i$ | $v_i$ |
|---|---|---|---|
| 1 | $(-1,0,0)$ | $(0,0,0)$ | $(0,0,0)$ |
| 2 | $(0,0,1)$ | $(0,L,0)$ | $(L,0,0)$ |
| 3 | $(0,0,0)$ | - | $(0,1,0)$ |
| 4 | $(-1,0,0)$ | $(0,2L,0)$ | $(0,0,2L)$ |
| 5 | $(0,0,0)$ | - | $(1,0,0)$ |
| 6 | $(0,1,0)$ | $(L,0,0)$ | $(0,0,L)$ |

(b) To determine the spatial velocity $\mathcal{V}_s$, we first need to find the space Jacobian $J_s$. Recall that the $i^{th}$ column of $J_s$ is the twist for joint axis $i$ expressed in the fixed frame, assuming the robot is at an arbitrary configuration $\theta$ rather than the zero position. Figure 1 shows the left arm at the given configuration $\theta = (\frac{\pi}{3}, 0, 0, -\frac{2\pi}{3}, L, 0)$. Denote the $i^{th}$ column of the space Jacobian by $\mathcal{V}_{si}(\theta) = (w_{si}, v_{si})$, where $v_{si} = -w_{si} \times q_{si}$. The corresponding Jacobian values are listed below:

| Column $i$ | $w_{si}$ | $q_{si}$ | $v_{si}$ |
|---|---|---|---|
| 1 | $(-1,0,0)$ | $(0,0,0)$ | $(0,0,0)$ |
| 2 | $(0,\frac{\sqrt{3}}{2},\frac{1}{2})$ | $(0,0,-\frac{2}{\sqrt{3}}L)$ | $(L,0,0)$ |
| 3 | $(0,0,0)$ | - | $(0,\frac{1}{2},-\frac{\sqrt{3}}{2})$ |
| 4 | $(-1,0,0)$ | $(0,L,-\sqrt{3}L)$ | $(0,\sqrt{3}L,L)$ |
| 5 | $(0,0,0)$ | - | $(1,0,0)$ |
| 6 | $(0,\frac{1}{2},\frac{\sqrt{3}}{2})$ | $(2L,L,-\sqrt{3}L)$ | $(\sqrt{3}L,-\sqrt{3}L,L)$ |

Given $\dot{\theta} = (1,1,1,1,1,1)^T$, we get

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} = \begin{bmatrix} -2 \\ \frac{\sqrt{3}+1}{2} \\ \frac{\sqrt{3}+1}{2} \\ (1+\sqrt{3})L + 1 \\ \frac{1}{2} \\ 2L - \frac{\sqrt{3}}{2} \end{bmatrix}.$$

The linear velocity of the {b}-frame origin expressed in {s}-frame coordinates, $\dot{p}_{sb}$, can be obtained from the translational velocity component $v_s$ of the spatial velocity $\mathcal{V}_s = (\omega_s, v_s)$ as follows:

$$v_s = \dot{p}_{sb} + \omega_s \times (-p_{sb}),$$

where $p_{sb} = (2L, \frac{3}{2}L, -\frac{\sqrt{3}}{2}L)^T$, from which it follows that

$$\dot{p}_{sb} = v_s + \omega_s \times p_{sb} = (1 - \frac{1}{2}L,\ \frac{1}{2} + L,\ -\frac{\sqrt{3}}{2} - (2+\sqrt{3})L)^T.$$

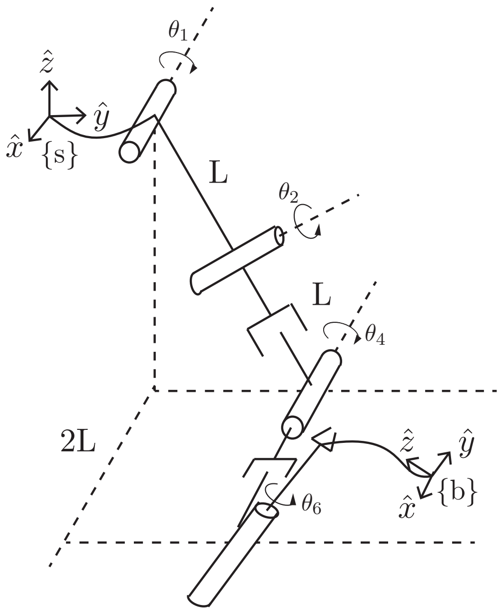



Figure 1: Groot's left arm at the given configuration.

(c) When $\theta_2$ is fixed to $\theta_2 = 0$, observe that only $\theta_5$ changes the $\hat{x}$-component of the {b}-frame location, i.e., $\theta_5 = p_x - L$. Now considering the orientation, since only $\theta_1, \theta_4, \theta_6$ affect the orientation of the end-effector, $R_{sb}$ can be expressed as follows:

$$\begin{aligned} R_{sb} &= \mathrm{Rot}(\omega_1, \theta_1) \cdot \mathrm{Rot}(\omega_4, \theta_4) \cdot \mathrm{Rot}(\omega_6, \theta_6) \\ &= \mathrm{Rot}(-\hat{x}, \theta_1) \cdot \mathrm{Rot}(-\hat{x}, \theta_4) \cdot \mathrm{Rot}(\hat{y}, \theta_6) \\ &= \mathrm{Rot}(\hat{x}, -\theta_1 - \theta_4) \cdot \mathrm{Rot}(\hat{y}, \theta_6). \end{aligned}$$

Since $R_{sg} = \mathrm{Rot}(\hat{x}, \alpha) \cdot \mathrm{Rot}(\hat{y}, \beta)$ is given, it follows that

$$\theta_1 + \theta_4 = -\alpha \pm 2\pi \ \text{ and } \theta_6 = \beta.$$

The remaining joint angles can be determined by examining the front view of the arm view (see

Figure 2):

$$
\begin{aligned}
\gamma &= -\theta_1 - \theta_4 = \alpha \mp 2\pi \;\rightarrow\; \cos\gamma = \cos\alpha,\ \sin\gamma = \sin\alpha \\
p_y &= (2L + \theta_3)\cos(-\theta_1) + L\cos\alpha \\
p_z &= (2L + \theta_3)\sin(-\theta_1) + L\sin\alpha \\
\theta_1 &= \operatorname{atan2}(L\sin\alpha - p_z, p_y - L\cos\alpha) \\
\theta_3 &= \begin{cases} \frac{L\sin\alpha - p_z}{\sin\theta_1} - 2L & \text{if } \theta_1 = \pm\pi/2 \\ \frac{p_y - L\cos\alpha}{\cos\theta_1} - 2L & \text{otherwise} \end{cases} \\
\theta_4 &= \begin{cases} -\alpha - \theta_1 + 2\pi & \text{if } -\alpha - \theta_1 \leqslant -\pi \\ -\alpha - \theta_1 - 2\pi & \text{else if } -\alpha - \theta_1 > \pi \\ -\alpha - \theta_1 & \text{otherwise.} \end{cases}
\end{aligned}
$$

At first glance the mechanism may appear to have both an elbow-up and elbow-down solution, but since the angle between the last link and the $\hat{y}$-axis is determined by $\alpha$, only one of these solutions is valid and it depends on the values of $\alpha$ and $atan2(p_y, p_x)$.

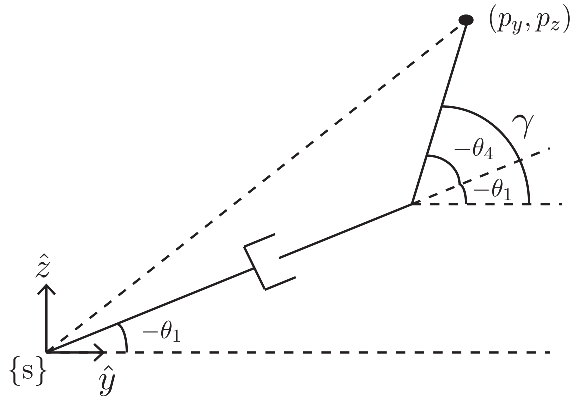



Figure 2: Front view.

**Problem 2**

(a) Observe that joints 1, 3, 4, and 6 are revolute, joint 5 is prismatic, and joint 2 is a helical (screw) joint. The respective joint twists are as follows:

$$
\begin{array}{lll}
\omega_1 = (1,0,0)^T, & q_1 = (0,0,1)^T, & h_1 = 0 \\
\omega_2 = (0,1,0)^T, & q_2 = (0,q_{2y},1)^T, & h_2 = 1 \\
\omega_3 = (0,0,1)^T, & q_3 = (0,2,q_{3z})^T, & h_3 = 0 \\
\omega_4 = (0,-1,0)^T, & q_4 = (0,q_{4y},-2)^T, & h_4 = 0 \\
\omega_5 = (0,0,0)^T, & v_5 = (0,0,1)^T, & h_5 = \infty \\
\omega_6 = (0,0,-1)^T, & q_6 = (0,0,-1)^T, & h_6 = 0,
\end{array}
$$

with $q_{2y}$, $q_{3z}$, $q_{4y}$ arbitrary. Therefore, exact $y$ coordinate of joint 2 and joint 4, and $z$ coordinate of joint 3 cannot be determined by using only screw information. According to the given matrix $M$, the end-effector is assumed to rotate about the $\hat{x}$-axis of the fixed frame by 180°. An example of a manipulator with the above kinematic parameters is shown in Figure 3.

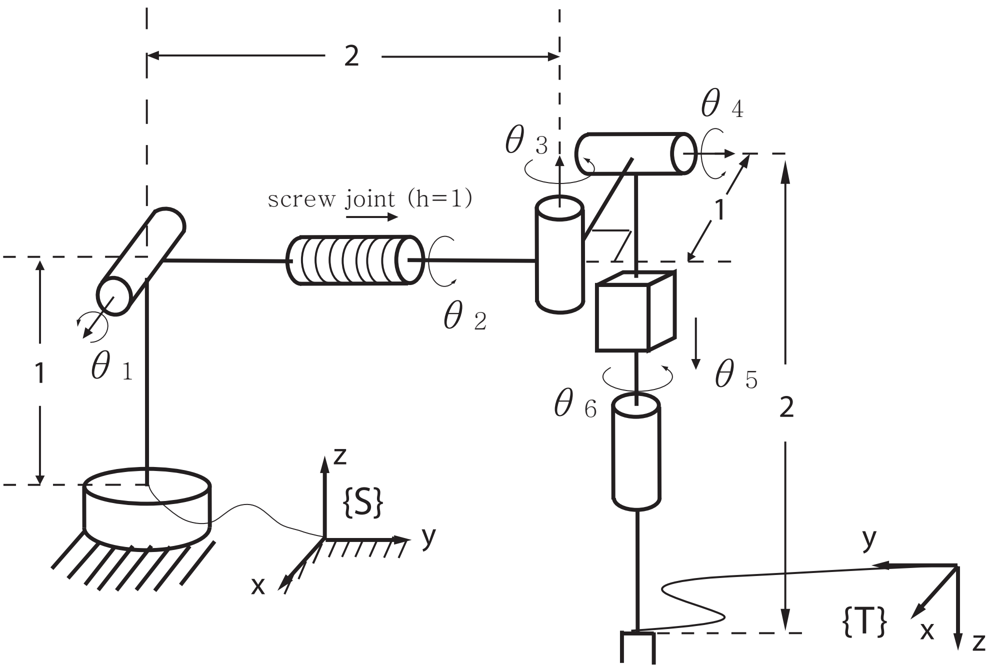



Figure 3: Manipulator for Problem 3

(b) The last three columns of the body Jacobian $J_b$ are as follows:

$$J_b(\theta) = \begin{bmatrix} & \sin\theta_6 & 0 & 0 \\ & -\cos\theta_6 & 0 & 0 \\ \cdots & 0 & 0 & -1 \\ & -(2+\theta_5)\cos\theta_6 & 0 & 0 \\ & -(2+\theta_5)\sin\theta_6 & 0 & 0 \\ & 0 & 1 & 0 \end{bmatrix}.$$

(c) For this problem one can use either the space Jacobian $J_s$ or the body Jacobian $J_b$. Evaluating $J_s$ in the zero position,

$$J_s(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 2 & -1 & 0 & 2 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}$$

Observe that columns 2, 3, 4, 5, and 6 are linearly dependent, since column 2 + column 3 − column 4 + column 5 = column 6. The zero position is therefore a kinematic singularity. Alternatively, evaluating $J_b$ in the zero position,

$$J_b(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & -2 & 0 & -2 & 0 & 0 \\ -2 & -1 & 1 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Observe that columns 2, 3, 4, 5, and 6 are linearly dependent, since column 2 + column 3 − column 4 + column 5 = column 6. Therefore the zero position is a kinematic singularity.

**Problem 3**
(a) Calculate the Jacobian by differentiating the forward kinematics

$$\begin{aligned} x &= L\cos\theta_2 + x_0 \\ y &= 3R + R\cos\theta_1 + L\sin\theta_2 + y_0, \end{aligned}$$

where $x_0$ and $y_0$ are constants depending on the choice of origin. Differentiating both sides leads to the following $2 \times 2$ Jacobian:

$$J(\theta) = \begin{bmatrix} 0 & -L\sin\theta_2 \\ -R\sin\theta_1 & L\cos\theta_2 \end{bmatrix}.$$

(b) Kinematic singularities occur when the Jacobian fails to be full rank; this happens when (i) $\sin\theta_1 = 0$, and (ii) $\sin\theta_2 = 0$. In the first case when $\sin\theta_1 = 0$, the Jacobian becomes

$$J = \begin{bmatrix} 0 & -L\sin\theta_2 \\ 0 & L\cos\theta_2 \end{bmatrix},$$

i.e., the manipulability ellipsoid in the end-effector velocity space collapses to the line in the direction $(-\sin\theta_2, \cos\theta_2)$. The end-effector therefore loses mobility in the direction orthogonal to this line, i.e., $(\cos\theta_2, \sin\theta_2)$. For the second case when $\sin\theta_2 = 0$, here the Jacobian becomes

$$J = \begin{bmatrix} 0 & 0 \\ -R\sin\theta_1 & L\cos\theta_2 \end{bmatrix},$$

from which it follows by inspection that the end-effector loses mobility along the line in the $(1, 0)$ direction.
(c) For convenience let $v = [\dot{x} \;\; \dot{y}]^T$, so that $v = J(\theta)\dot{\theta}$. Then assuming $J(\theta)$ is not singular, the constraint $1 = \dot{\theta}^T\dot{\theta} = v^T(J(\theta)J(\theta)^T)^{-1}v$ is the equation for an ellipse in the space of velocities $v \in \mathbb{R}^2$. At $\theta_1 = \theta_2 = \frac{\pi}{2}$, $J(\theta)$ becomes

$$J = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

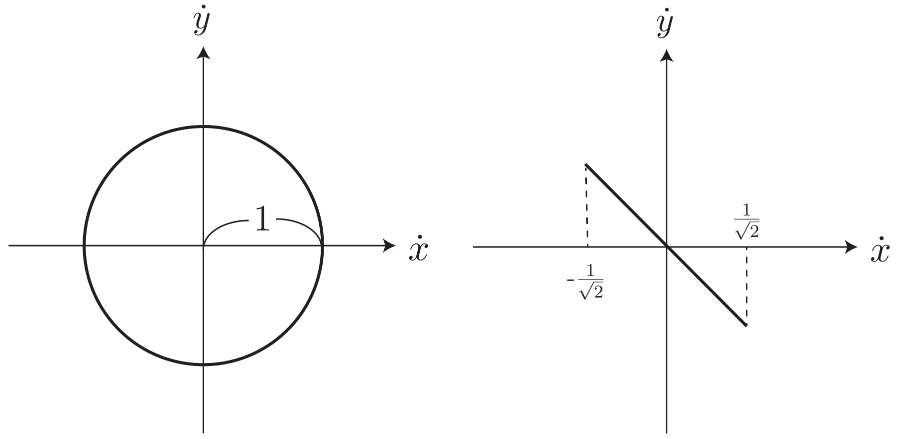
from which it can be easily verified that $JJ^T$ is the $2 \times 2$ identity matrix, and the corresponding manipulability ellipsoid is a unit circle. In the second case when $\theta_1 = 0$, $\theta_2 = \frac{\pi}{4}$, the Jacobian becomes

$$J = \begin{bmatrix} 0 & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix},$$

which is singular; the end-effector can only move along the line $\dot{x} + \dot{y} = 0$.
(d) The optimal design is obtained by minimizing $\mu_1$ with respect to the design parameters $R$, $L$ at the given configuration. The Jacobian $J$ at this configuration is

$$J = \begin{bmatrix} 0 & \frac{-L}{\sqrt{2}} \\ \frac{-R}{\sqrt{2}} & \frac{L}{\sqrt{2}} \end{bmatrix}, \quad JJ^T = \begin{bmatrix} \frac{L^2}{2} & -\frac{L^2}{2} \\ -\frac{L^2}{2} & \frac{L^2+R^2}{2} \end{bmatrix}.$$

We now recall some basic properties of matrics and their eigenvalues: given a matrix $A \in \mathbb{R}^{n\times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, recall that $\mathrm{tr}(A) = \lambda_1 + \ldots + \lambda_n$ and $\det(A) = \lambda_1 \cdots \lambda_n$. Moreover




(a) (b)

if $A$ is symmetric all its eigenvalues are real; if $A$ is additionally positive-semidefinite, then all its eigenvalues are non-negative. Since $JJ^T$ is symmetric positive-semidefinite, both its eigenvalues are real and non-negative; we label these two eigenvalues as $\lambda_{\max}$ and $\lambda_{\min}$. Then from the above matrix eigenvalue properties we have

$$\begin{aligned}\lambda_{\max}\lambda_{\min} &= \frac{L^2R^2}{4} \\ \lambda_{\max}+\lambda_{\min} &= \frac{2L^2+R^2}{2}.\end{aligned}$$

Noting that $\lambda_{\max} \geq \lambda_{\min} \geq 0$, the above can be solved to obtain

$$\frac{\lambda_{\max}}{\lambda_{\min}} = \frac{2+(R/L)^2+\sqrt{4+(R/L)^4}}{2+(R/L)^2-\sqrt{4+(R/L)^4}}.$$

(Here's a quick calculation: setting $c = L^2R^2/4$ and $b = (2L^2+R^2)/2$, one can write $\lambda_{\max}\lambda_{\min} = c$ and $\lambda_{\max}+\lambda_{\min} = b$. Then $\lambda_{\max} = c/\lambda_{\min}$, $\lambda_{\min} = c/\lambda_{\max}$, and $\lambda_{\max}+(c/\lambda_{\max}) = \lambda_{\min}+(c/\lambda_{\min}) = b$, from which it follows that $\lambda_{\max}$ and $\lambda_{\min}$ are roots of the same quadratic $\lambda^2 - b\lambda + c = 0$. Since $\lambda_{\max} \geq \lambda_{\min}$, it follows that

$$\frac{\lambda_{\max}}{\lambda_{\min}} = \frac{b+\sqrt{b^2-4c}}{b-\sqrt{b^2-4c}}$$

as claimed.) Setting $r = R/L$ and $x = r^2$, the ensuing optimization simplifies to

$$\min_x \frac{2\sqrt{4+x^2}}{2+x-\sqrt{4+x^2}}.$$

Setting the derivative of the above objective function to zero and solving for $x$ should, after some calculation, lead to the solution $x^* = 2$. Alternatively, the following substitutions simplify matters somewhat: set $t = \sqrt{4+x^2}$ and $\frac{dt}{dx} = t'$, so that the minimization becomes

$$\min_x \frac{2t}{2+x-t}$$

subject to $x = tt'$. Let $x^*$ be the minimizer and $t^* = \sqrt{4+x^{*2}}$. Then

$$\frac{d}{dx}\frac{2t}{(2+x-t)^2} = \frac{1}{(2+x-t)^2}(2t'(2+x-t)-2t(1-t')) = 0 \;\text{ at }\; x = x^*,$$

or equivalently, $4t'^* + 2t'^* x^* - 2t^* = 0$. Multiplyiing both sides by $t^*$,

$$4x^* + 2x^{*2} - 2(4 + x^{*2}) = 0,$$

so that $x^* = 2$ and $r^* = \sqrt{2}$. The optimal design is therefore achieved when $\frac{R}{L} = \sqrt{2}$.
(e) Here we maximize $\mu_2$ with respect to the joint angles. From the general expression for $J$ obtained in (a),

$$JJ^T = \begin{bmatrix} L^2 \sin^2\theta_2 & -L^2 \sin\theta_2 \cos\theta_2 \\ -L^2 \sin\theta_2 \cos\theta_2 & R^2 \sin^2\theta_1 + L^2 \cos^2\theta_2 \end{bmatrix}.$$

Setting $L = R = 1$, we have $\det(JJ^T) = \sin^2\theta_1 \sin^2\theta_2$, which is maximized when $\sin^2\theta_1 \sin^2\theta_2 = 1$. Therefore, $\theta_1 = \frac{\pi}{2} + n\pi,\ \theta_2 = \frac{\pi}{2} + m\pi$, where $n, m$ are integers.

**Problem 4**
(a) Writing out $\mathcal{F}_b = [\mathrm{Ad}_{T_{ab}}]^T \mathcal{F}_a$,

$$\begin{aligned} m_b &= R_{ba} m_a - R_{ba}[p_{ab}] f_a \\ f_b &= R_{ab} f_a. \end{aligned}$$

Repackaging these into Roy's form, we get

$$\begin{aligned} \mathcal{F}'_b &= \begin{bmatrix} R_{ba} & 0 \\ -R_{ba}[p_{ab}] & R_{ba} \end{bmatrix} \mathcal{F}'_a \\ &= \begin{bmatrix} R_{ba} & 0 \\ R_{ba}[R_{ba}^T p_{ba}] & R_{ba} \end{bmatrix} \mathcal{F}'_a \\ &= \begin{bmatrix} R_{ba} & 0 \\ [p_{ba}] R_{ba} & R_{ba} \end{bmatrix} \mathcal{F}'_a \\ &= [\mathrm{Ad}_{T_{ba}}] \mathcal{F}'_a, \end{aligned}$$

so Roy is correct.
(b) Using the transformation rule for $6 \times 6$ spatial mass matrices,

$$\mathcal{G}_a = [\mathrm{Ad}_{T_{ba}}]^T \mathcal{G}_b [\mathrm{Ad}_{T_{ba}}] = \begin{bmatrix} R_{ba}^T \mathcal{I}_b R_{ba} + R_{ba}^T [p_{ba}]^T \mathfrak{m} [p_{ba}] R_{ba} & R_{ba}^T [p_{ba}]^T \mathfrak{m} R_{ba} \\ R_{ba}^T [p_{ba}]^T \mathfrak{m} R_{ba} & R_{ba}^T \mathfrak{m} R_{ba} \end{bmatrix}.$$

From the top-left entry of the above equality,

$$\mathcal{I}_a = R_{ba}^T \mathcal{I}_b R_{ba} + R_{ba}^T [p_{ba}]^T \mathfrak{m} [p_{ba}] R_{ba} = R_{ab} \mathcal{I}_b R_{ab}^T + [p_{ab}]^T \mathfrak{m} [p_{ab}].$$

(c) Calculating $[\mathcal{V}_i] = \dot{T}_{0i} T_{0i}^{-1}$,

$$\begin{aligned} [\mathcal{V}_i] &= \dot{T}_{0i} T_{0i}^{-1} \\ &= [\mathcal{S}_1]\dot{\theta}_1 + e^{[\mathcal{S}_1]\theta_1} [\mathcal{S}_2] e^{-[\mathcal{S}_1]\theta_1} \dot{\theta}_2 + \ldots + e^{[\mathcal{S}_1]\theta_1} \ldots e^{[\mathcal{S}_{i-1}]\theta_{i-1}} [\mathcal{S}_i] e^{-[\mathcal{S}_{i-1}]\theta_{i-1}} \ldots e^{-[\mathcal{S}_1]\theta_1} \dot{\theta}_i \\ &= [\mathcal{V}_{i-1}] + e^{[\mathcal{S}_1]\theta_1} \ldots e^{[\mathcal{S}_{i-1}]\theta_{i-1}} [\mathcal{S}_i] e^{-[\mathcal{S}_{i-1}]\theta_{i-1}} \ldots e^{-[\mathcal{S}_1]\theta_1} \dot{\theta}_i, \end{aligned}$$

from which it follows that

$$\mathcal{V}_i = \mathcal{V}_{i-1} + \mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1} \ldots e^{[\mathcal{S}_{i-1}]\theta_{i-1}}} (\mathcal{S}_i) \dot{\theta}_i.$$

**M2794.002700 Introduction to Robotics**
**Final Examination**
**6:30-9:30 PM, June 14, 2018**
**CLOSED BOOK, CLOSED NOTES**

**Problem 1** (30 points)
(a) Spot the robot dog is grasping a door handle as show in Figure 1. Modeling each of Spot's feet as spherical joints connected to ground, use Grübler's formula to determine Spot's overall degrees of freedom.

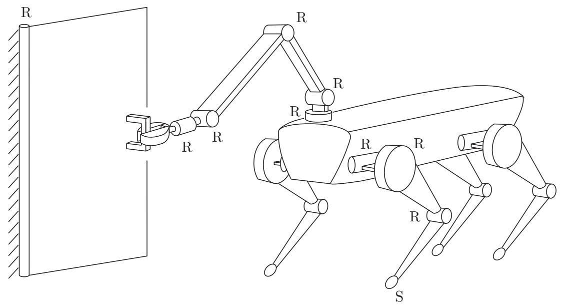
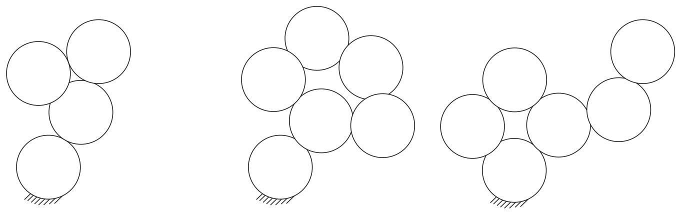



Figure 1: Problem 1(a): Spot grasping a door handle.

(b) The steel magnetic marbles shown in Figure 2(a) are all of the same size and only attract each other (no repelling forces). Two marbles in contact can be modelled as a "magnetic joint"—the marbles can slide and rotate without friction while maintaining contact with each other. Determine the degrees of freedom of a magnetic joint, and use Grübler's formula to find the degrees of freedom of the connected marbles in Figure 2(a). (Note that the bottom marble is rigidly attached to ground.)
(c) The marbles can be connected in many different ways; for example, Figure 2(b) shows two possible arrangements using six marbles. Using exactly six marbles, find the arrangements that result in the **minimum** and **maximum** degrees of freedom (note that one of the marbles must be rigidly attached to ground). Draw the minimum and maximum dof arrangements and explain your answer.



(a) (b)

Figure 2: Magnetic marble arrangements for Problems 1(b)-(c).

**Problem 2** (30 points)
For trees like Figure 3(a) that are unstable, adding support structures can make them more stable.
(a) Modeling the tree as a planar rigid object as shown in Figure 3(b), suppose we add supports at four points A, B, C, D as shown. Modeling each of the supports as frictionless point contacts, determine the range of values of $d$ for point contact D so that the tree is form closure.
(b) Suppose a very heavy fruit is attached to the tree at point Q as shown in Figure 3(c). Are three supports A, B, C (again modelled as frictionless point contacts) enough to keep the tree from falling over?

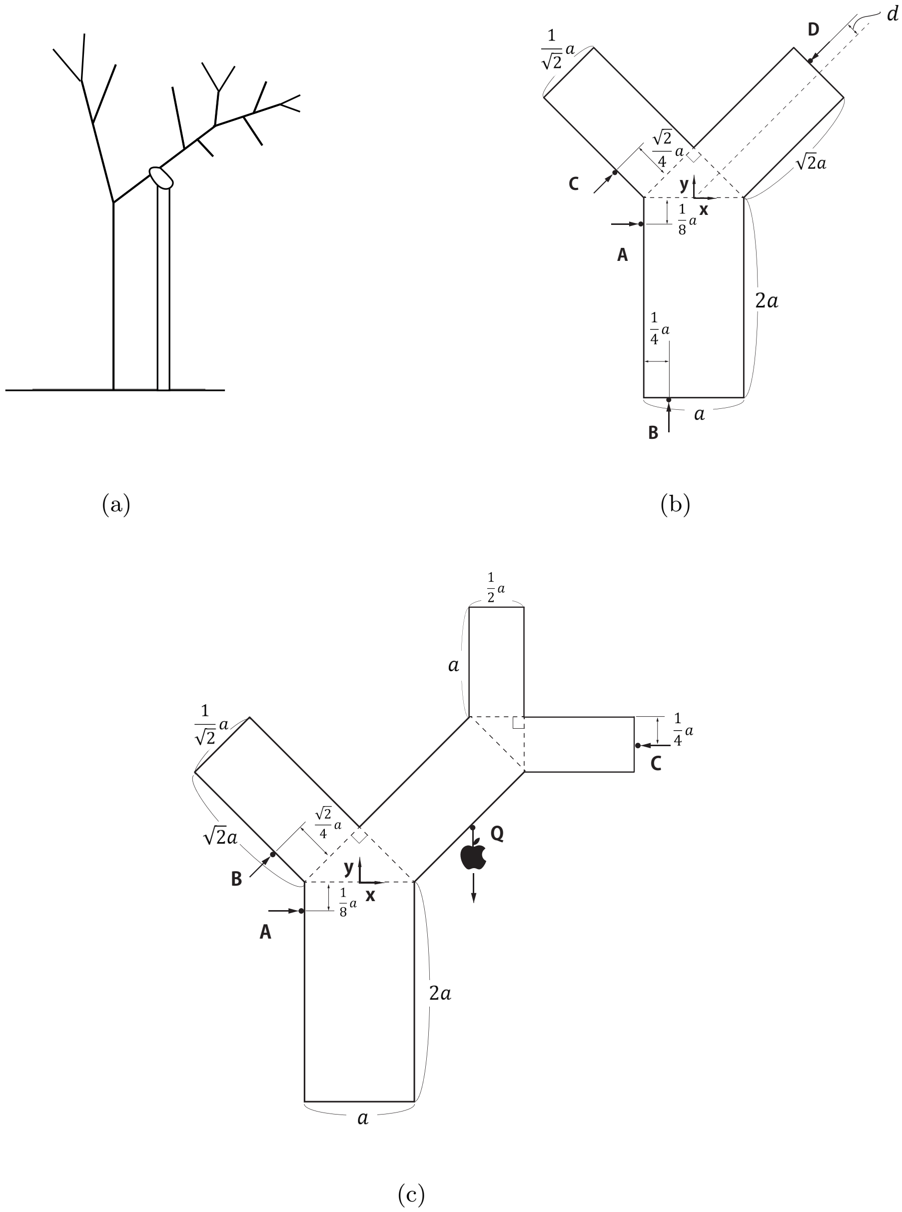



Figure 3: Figures for Problem 2.

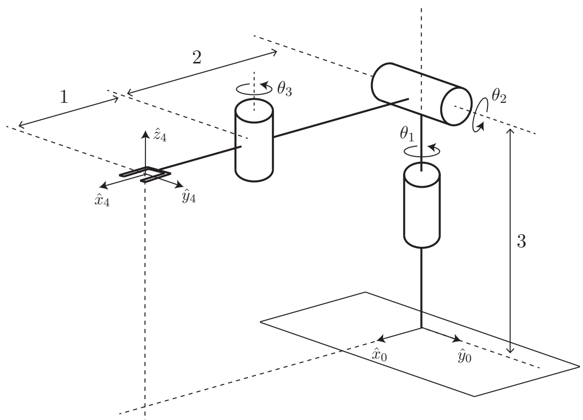
**Problem 3** (60 points)
(a) For the 3R robot of Figure 4(a) shown in its zero position, assign appropriate link frames and derive the Denavit-Hartenberg parameters. You may ignore the frames shown in the figure and draw your own fixed and link frames.
(b) For the 3R robot of Figure 4(a) shown in its zero position, derive the forward kinematics in the following product-of-exponentials form:

$$T_{04} = M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_3]\theta_3},$$

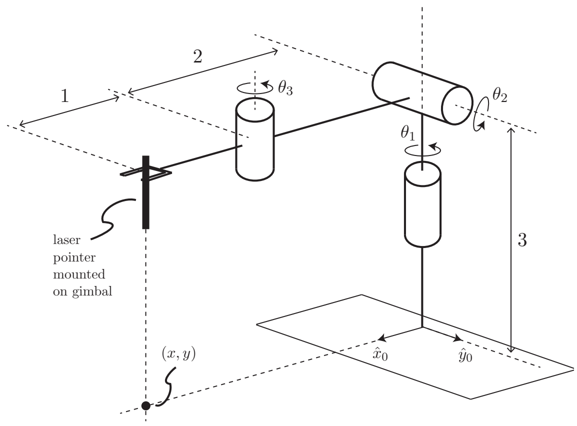
where $M \in \mathrm{SE}(3)$ and $\mathcal{B}_i \in \mathrm{se}(3)$, $i = 1, 2, 3$.
(c) Calculate the volume of all points in $\mathbb{R}^3$ reachable by the robot tip (take the frame $\{4\}$ origin to be the robot tip).
(d) Now suppose the robot is grasping a laser pointer mounted on a gimbal, so that the laser pointer always remains vertical (that is, normal to the $x$-$y$ plane of the fixed frame). Derive the forward kinematic mapping from $(\theta_1, \theta_2, \theta_3)$ to the point $(x, y)$ on the plane indicated by the laser pointer.
(e) Derive the Jacobian $J(\theta) \in \mathbb{R}^{2\times 3}$ relating the joint rates $\dot{\theta} = (\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3)^T$ to the velocity $v = (\dot{x}, \dot{y})^T$ of the laser pointer. Show that $\dot{\theta} = J^T(JJ^T)^{-1}v$ is a valid inverse velocity kinematics solution as long as the inverse exists.
(f) Suppose the robot is in its initial position, and we wish to move the laser pointer along a linear path from some initial point $P_0 \in \mathbb{R}^2$ to final point $P_1 \in \mathbb{R}^2$. Using the solution $\dot{\theta} = J^T(JJ^T)^{-1}v$, derive an iterative numerical procedure for solving this inverse kinematics problem.
(g) More generally, show that, assuming $JJ^T$ is invertible,

$$\dot{\theta} = J^T(JJ^T)^{-1}v + \left(I - J^T(JJ^T)^{-1}J\right)\lambda,$$

is a solution to $v = J(\theta)\dot{\theta}$ for any $\lambda \in \mathbb{R}^3$. Can you attach any special physical or other meaning to the particular solution $\dot{\theta} = J^T(JJ^T)^{-1}v$?




(a) 3-dof arm for Problem 3(a)-(c).




(b) 3-dof arm holding laser pointer for Problem 3(d)-(f).

Figure 4: Figures for Problem 3.

**Problem 4** (30 points)
(a) A six-dof spatial open chain has two of its prismatic joint axes coplanar, and two of its revolute joints axes normal to the plane spanned by the two prismatic joint axes (see Figure 5(a)). Is this configuration singular? You must explain your answer to receive full credit.
(b) A six-dof spatial open chain has five of its revolute joint axes intersecting a common line, and a prismatic joint axis perpendicular to the common line (see Figure 5(b)). Is this configuration singular? You must explain your answer to receive full credit.
(c) The spatial RRRRRP open chain of Figure 6 is shown in its zero position. Find at least three different singular configurations for this robot, and explain why these configurations are singular in terms of the joint axis screws.
(d) At its zero position, can the robot resist arbitrary wrenches applied to the tip? If yes, can the robot generate any arbitrary wrench at the tip as well? Explain your answer.

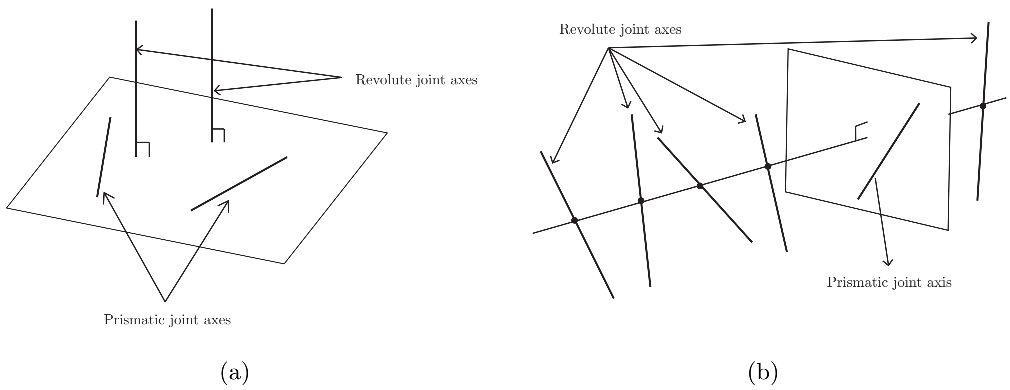



Figure 5: Two six-dof spatial chains for Problem 4(a)-(b).

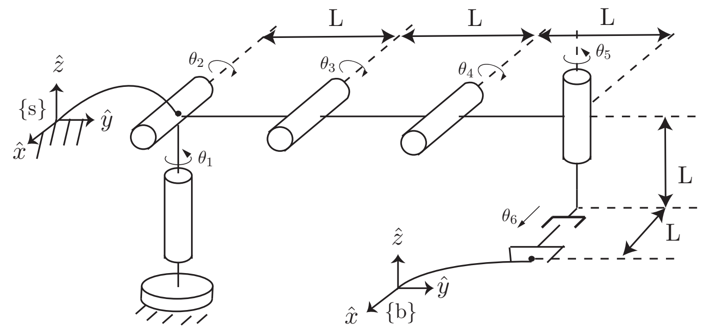



Figure 6: Spatial RRRRRP open chain for Problem 4(c)-(d).

**Problem 5** (30 points)
(a) Consider a one-dof mass-spring-damper system with dynamics

$$\ddot{x} + 2\dot{x} + x = f,$$

Given arbitrary initial conditions $x(0)$ and $\dot{x}(0)$, find all PD controls $f$ of the form $f = -k_p x - k_d \dot{x}$ that will drive the system to rest as fast as possible.
(b) Recall that the dynamics of an $n$-dof open chain robot can be written in the form

$$\tau = M(\theta)\ddot{\theta} + b(\theta, \dot{\theta}).$$

Assume there is no gravity, let $n = 2$, and suppose we wish to experimentally estimate the mass matrix $M(0) \in \mathbb{R}^{2\times 2}$ at the zero position $\theta = 0$. We perform the following steps: (i) set the robot at rest (zero velocities and accelerations) in its zero position; (ii) apply an input joint torque $\tau \in \mathbb{R}^2$ to the robot; (iii) measure the corresponding joint acceleration vector $\ddot{\theta} \in \mathbb{R}^2$. Which of the measurement pairs $\tau \mapsto \ddot{\theta}$ listed below are valid, and which are invalid? Determine $M(0)$ for all valid pairs. You must explain all your answers to receive full credit.

1. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}.$
2. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$
3. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 8 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$

(c) After repeating the above experiment twice, because of noisy measurements, two valid estimates for $M(0)$, $\tilde{M}_1$ and $\tilde{M}_2$, are obtained:

$$\tilde{M}_1 = \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}, \quad \tilde{M}_2 = \begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix}.$$

Suggest a sensible way to calculate the average of these two matrices (*Hint*: it may be useful to think about eigenvalues).

**Problem 6** (30 points)
(a) Given two rotations $R_0, R_1 \in \mathrm{SO}(3)$, the shortest path $R(s)$ connecting $R_0$ to $R_1$ is given by

$$R(s) = R_0 e^{[r]s}, \quad [r] = \log(R_0^{-1} R_1),$$

where $[r] \in \mathrm{so}(3)$ is chosen so that $\|r\| \leq \pi$, and $R(0) = R_0$, $R(1) = R_1$. The length of this shortest path can then be calculated to be $\|r\|$. Under the reference frame transformation $R_0 \mapsto R_0' = R_0 P$, $R_1 \mapsto R_1' = R_1 P$ for some constant $P \in \mathrm{SO}(3)$, show that the distance between $R_0'$ and $R_1'$ does not change.
(b) Let $R(s)$, $s \in [0, 1]$, be the Bézier curve in SO(3) generated by the three "control points" $R_0 = I$, $R_1 = \mathrm{Rot}(\hat{x}, \pi/2)$, $R_2 = \mathrm{Rot}(\hat{z}, \pi/2)$, such that $R(0) = R_0$ and $R(1) = R_2$. Derive the algorithm for computing $R(s)$, and calculate $R(0.5)$.

**M2794.0027 Introduction to Robotics**
**Final Examination Solutions**
**June 14, 2018**
**CLOSED BOOK, CLOSED NOTES**

**Problem 1** (30 points)
(a) Applying the spatial version of Grübler's formula,

- $N = 18$ (links) $+1$ (ground) $= 19$;
- $J = 4$ (S joints) $+$ 18 (R joints) $= 22$;
- $\Sigma f_i = 3 \times 4$ (S joints) $+1 \times 18$ (R joints) $= 30$;
- $\text{dof} = 6(N - 1 - J) + \Sigma f_i = 6(19 - 1 - 22) + 30 = 6$.

(b) A magnetic joint allows two marbles to slide and rotate while maintaining a fixed distance between their centers. Since this is a single equality constraint between two rigid bodies, a magnetic joint has $6 - 1 = 5$ degrees of freedom. Applying the spatial version of Grübler's formula to the marble arrangement then leads to

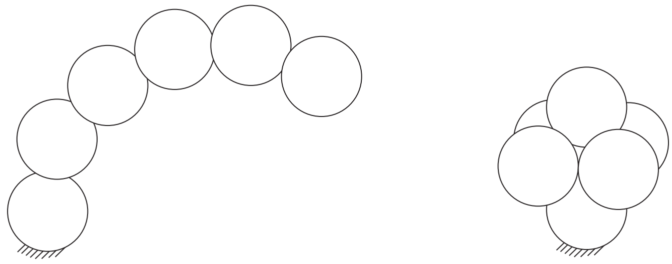
- $N = 3$ (links) $+1$ (ground) $= 4$;
- $J = 4$ (magnetic joints);
- $\Sigma f_i = 5 \times 4$ (magnetic joints) $= 20$;
- $\text{dof} = 6(N - 1 - J) + \Sigma f_i = 6(4 - 1 - 4) + 20 = 14$.

(c) Observing that the number of links is fixed, the degrees of freedom only depends on the number of joints (or equivalently, the number of contacts between marbles); the more the contacts, the lower the overall mobility. It thus seems reasonable to find the arrangements with the minimum and the maximum number of contacts. Quite clearly the serially connected marbles as shown in Figure 1(a) will have maximum dof:

- $N = 5$ (links) $+1$ (ground) $= 6$;
- $J = 5$ (magnetic joints);
- $\Sigma f_i = 5 \times 5$ (magnetic joints) $= 25$;
- $\text{dof} = 6(N - 1 - J) + \Sigma f_i = 6(6 - 1 - 5) + 25 = 25$.

The minimum dof case occurs when the six marbles (or more specifically, their centers) form an octahedron as shown in Figure 1(b):

- $N = 5$ (links) $+1$ (ground) $= 6$;
- $J = 12$ (magnetic joints);
- $\Sigma f_i = 5 \times 12$ (magnetic joints) $= 60$;
- $\text{dof} = 6(N - 1 - J) + \Sigma f_i = 6(6 - 1 - 12) + 60 = 18$.



(a)

(b)

Figure 1: Maximum and minimum dof marble arrangements for Problem 1(c).

**Problem 2** (30 points)
(a) Express the static equilibrium force closure conditions in the standard linear form $Ax = b$, where $A \in \mathbb{R}^{3\times 4}$ is specified, and the objective is to determine whether a nonnegative solution $x \geq 0$ exists for any arbitrary $b \in \mathbb{R}^3$. Deriving $A$ and performing Gauss-Jordan elimination leads to the following matrix:

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ a/8 & -a/4 & -a & \sqrt{2}d \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{8\sqrt{2}}{7}\frac{d}{a} - \frac{8}{7} \\ 0 & 1 & 0 & \frac{8\sqrt{2}}{7}\frac{d}{a} - \frac{8}{7} \\ 0 & 0 & 1 & \frac{1}{7} - \frac{8\sqrt{2}}{7}\frac{d}{a} \end{bmatrix}.$$

The grasp is force closure if and only if the fourth column has all elements negative, which leads to the following two inequalities:

$$\frac{8\sqrt{2}}{7}\frac{d}{a} - \frac{8}{7} < 0$$
$$\frac{1}{7} - \frac{8\sqrt{2}}{7}\frac{d}{a} < 0.$$

Thus, $\frac{a}{8\sqrt{2}} < d < \frac{a}{2\sqrt{2}}$ must be satisfied for the contacts to be force closure.
(b) To keep the tree stationary, the sum of all the forces and moments exerted on the tree must be zero:

$$\sum_i f_i = 0 \text{ and } \sum_i \tau_i = 0.$$

Let $f_a, f_b, f_c$ respectively be the forces at contact points A, B, and C, and let $F_Q$ be the force at Q. The problem can then be reformulated as follows: Does there exist $a_i \geq 0 (i = 1, 2, 3)$ such that

$$\begin{bmatrix} 1 \\ 0 \\ \frac{a}{8} \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ 1 \\ -a \end{bmatrix} a_2 + \begin{bmatrix} -1 \\ 0 \\ \frac{5}{4}a \end{bmatrix} a_3 + \begin{bmatrix} 0 \\ -1 \\ -x \end{bmatrix} b_Q = 0,$$

where $\frac{a}{2} \leq x \leq 3a$. From the middle equation we obtain $a_2 = b_Q$, while from the first we have $a_1 = a_3 - b_Q$. The third equation lead to

$$\frac{a}{8}a_1 - aa_2 + \frac{5a}{4}a_3 - xb_Q = 0, \text{ for all } b_Q \geq 0.$$

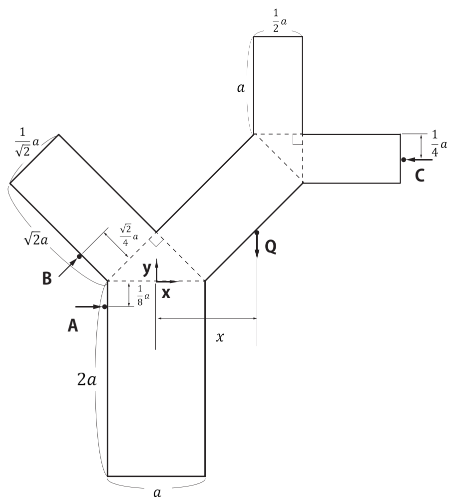



Figure 2: Three point contacts to resist the external force.

Rearranging the above leads to the following solution:

$$a_1 = \left(\frac{8}{11}\frac{x}{a} - \frac{2}{11}\right) b_Q \geq 0$$
$$a_2 = b_Q \geq 0$$
$$a_3 = \left(\frac{9}{11} + \frac{8}{11}\frac{x}{a}\right) b_Q \geq 0.$$

This grasp is therefore able to resist the given external force.

**Problem 3** (60 points)
(a) Link reference frames for finding the Denavit-Hartenberg parameters can be attached as shown in Figure 3 (Other choices are of course possible). The corresponding D-H parameters are as follows:

| $i$ | $\alpha_i$ | $a_i$ | $d_i$ | $\phi_i$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 3 | $\theta_1$ |
| 2 | $-90°$ | 0 | 0 | $\theta_2$ |
| 3 | $90°$ | 2 | 0 | $\theta_3$ |
| 4 | 0 | 1 | 0 | 0 |

(b) Writing the forward kinematics in the form

$$T_{04} = M e^{[\mathcal{B}_1]\theta_1} e^{[\mathcal{B}_2]\theta_2} e^{[\mathcal{B}_3]\theta_3},$$

then $M \in SE(3)$ is the end-effector frame at the zero position, which by inspection is

$$M = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

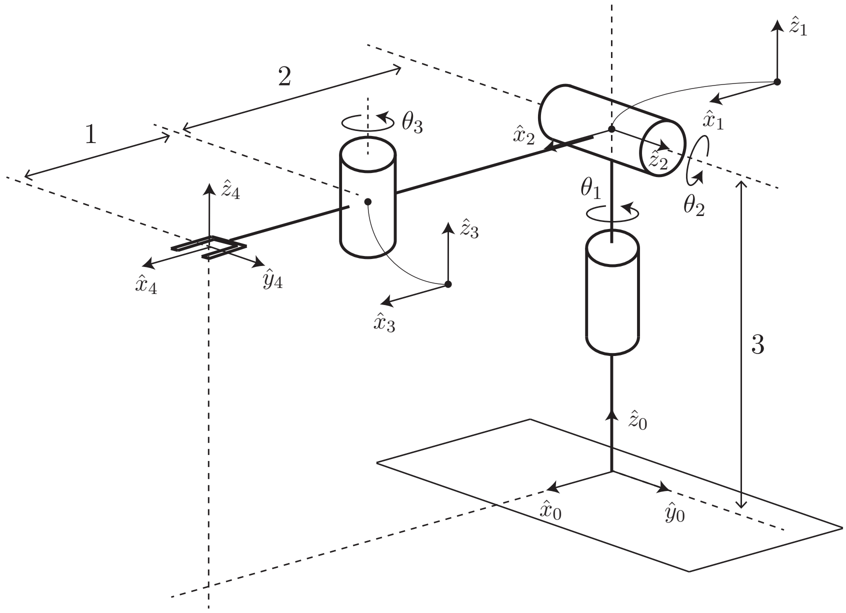



Figure 3: Reference frames for Denavit-Hartenberg parameters

The $\mathcal{B}_i = (\omega_i, v_i)$, $i = 1, 2, 3$ are the joint axis screws expressed in terms of the end-effector frame $\{4\}$ at the zero position; by inspection they are

$$\begin{array}{rclrcr} \omega_1 & = & (0,0,1) & v_1 & = & (0,3,0) \\ \omega_2 & = & (0,1,0) & v_2 & = & (0,0,-3) \\ \omega_3 & = & (0,0,1) & v_3 & = & (0,1,0) \end{array}.$$

(c) The problem is asking for the Cartesian workspace volume of the end-effector frame; that is the volume of the set of all points reachable by the origin of the end-effector frame $\{4\}$. The Cartesian positioning workspace is a spherical shell with inner radius 1 and outer radius 3; the corresponding volume is

$$\text{Volume} = \frac{4}{3}\pi(3^3 - 1^3) = \frac{104}{3}\pi.$$

(d) The method requiring the least calculation is probably to attach a moving frame $\{2\}$ to link 2: place its origin at the intersection of joint axes 1 and 2, with its $\hat{y}_2$-axis aligned with joint axis 2 and $\hat{x}_2$-axis aligned with the link of length 2. Then abbreviating $\sin\theta_1 = s_1$, $\cos\theta_2 = c_2$, etc.,

$$R_{02} = \text{Rot}(\hat{z}, \theta_1) \cdot \text{Rot}(\hat{y}, \theta_2) = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 \\ s_1 c_2 & c_1 & s_1 s_2 \\ -s_2 & 0 & c_2 \end{bmatrix},$$

and since the end-effector Cartesian position $p$ is given by

$$p = 3\hat{z}_0 + (2 + c_3)\hat{x}_2 + s_3\hat{y}_2,$$

with

$$\begin{array}{rcl} \hat{x}_2 & = & c_1 c_2 \hat{x}_0 + s_1 c_2 \hat{y}_0 - s_2 \hat{z}_0 \\ \hat{y}_2 & = & -s_1 \hat{x}_0 + c_1 \hat{y}_0 \\ \hat{z}_2 & = & c_1 s_2 \hat{x}_0 + s_1 s_2 \hat{y}_0 + c_2 \hat{z}_0, \end{array}$$

it follows that

$$\begin{aligned} x &= 2c_1c_2 + c_1c_2c_3 - s_1s_3 \\ y &= 2s_1c_2 + s_1c_2c_3 + c_1s_3. \end{aligned}$$

(e) Differentiating $x$ and $y$ with respect to time $t$, the Jacobian $J(\theta)$ can be obtained as

$$J(\theta) = \begin{bmatrix} -s_1c_2(2+s_3) + c_1s_3 & -c_1s_2(2+c_3) & -c_1c_2s_3 - s_1c_3 \\ c_1c_2(2+c_3) - s_1s_3 & -s_1s_2(2+c_3) & -s_1c_2s_3 + c_1c_3 \end{bmatrix}.$$

Verifying that $\dot{\theta} = J^T(JJ^T)^{-1}v$ is a valid inverse velocity kinematics solution is trivial: just substitute the above expression for $\dot{\theta}$ into $v = J(\theta)\dot{\theta}$ to get the identity $v = v$.
(f) Parametrizing the linear path from $P_0$ to $P_1$ as $P(s) = sP_1 + (1-s)P_0$, the simplest time-scaling function $s(t)$ is to just take $s = t$ (although such a trajectory would result in non-zero initial and final velocities, which for practical reasons is undesirable). Then $v = \dot{P} = P_1 - P_0$, and the simplest iterative joint-space inverse kinematics solution is given by the usual

$$\theta(t+\Delta t) = \theta(t) + \Delta t J^T(\theta(t)) \left(J(\theta(t))J^T(\theta(t))\right)^{-1}(P_1 - P_0).$$

(g) As was the case with part (e), substituting

$$\dot{\theta} = J^T(JJ^T)^{-1}v + \left(I - J^T(JJ^T)^{-1}J\right)\lambda,$$

into $v = J(\theta)\dot{\theta}$ should trivially verify that the above $\dot{\theta}$ is a valid solution for all $\lambda$. As to what physical meaning can be associated to the solution corresponding to $\lambda = 0$, let's examine $\|\dot{\theta}\|^2$ more closely: following a simple calculation, it can be observed that

$$\|\dot{\theta}\|^2 = \|J^T(JJ^T)^{-1}v\|^2 + \|\left(I - J^T(JJ^T)^{-1}J\right)\lambda\|^2.$$

What this means is that $\dot{\theta} = J^T(JJ^T)^{-1}v$ is the minimum-norm solution, i.e., among the many possible solutions $\dot{\theta}$ (represented by different choices of $\lambda$), this is the one that minimizes $\|\dot{\theta}\|^2$.

**Problem 4** (30 points)
(a) Without loss of generality, label the two prismatic joint axes as 1 and 2, and the two revolute joint axes as 3 and 4. Choose a fixed frame such that the prismatic joint axes all lie on the $x$-$y$ plane. Then

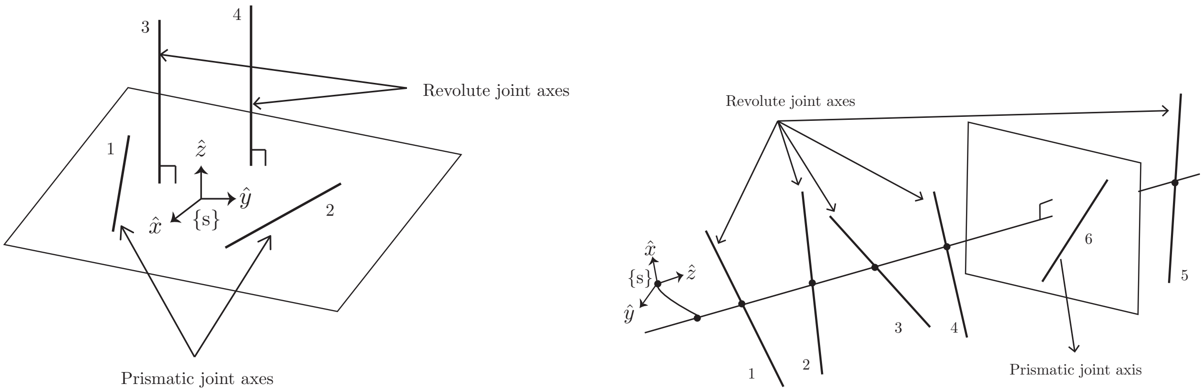
$$J_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_{1y} \\ v_{1z} \end{bmatrix}, \quad J_{s2} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_{2y} \\ v_{2z} \end{bmatrix}, \quad J_{s3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ v_{3y} \\ v_{3z} \end{bmatrix}, \quad J_{s4} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ v_{4y} \\ v_{4z} \end{bmatrix}.$$

These four columns cannot be linearly independent since they only have three nonzero components. Thus, this configuration is singular.
(b) Without loss of generality, label the revolute joint axes from 1 to 5, and the prismatic joint axis as 6. Choose a fixed frame such that the common line is along its $z$-axis, and select the intersection between this common line and joint axis $i$ as the reference point $q_i \in \mathbb{R}^3$, $i = 1, \cdots, 5$. Each $q_i$ is thus of the form $q_i = (0, 0, q_{iz})$, and $v_i = -w_i \times q_i = (w_{iy}q_{iz}, -w_{iz}q_{iz}, 0) = (v_{ix}, v_{iy}, 0)$. Since

the prismatic joint axis is perpendicular to the common line, the $z$-component of $v_6$ is zero, i.e., $v_6 = (v_{6x}, v_{6y}, 0), w_6 = (0, 0, 0)$. The space Jacobian $J_s(\theta)$ thus becomes

$$\begin{bmatrix} w_{1x} & w_{2x} & w_{3x} & w_{4x} & w_{5x} & 0 \\ w_{1y} & w_{2y} & w_{3y} & w_{4y} & w_{5y} & 0 \\ w_{1z} & w_{2z} & w_{3z} & w_{4z} & w_{5z} & 0 \\ v_{1x} & v_{2x} & v_{3x} & v_{4x} & v_{5x} & v_{6x} \\ v_{1y} & v_{2y} & v_{3y} & v_{4y} & v_{5y} & v_{6y} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which is clearly singular.




(a) (b)

Figure 4: Two six-dof spatial chains for Problem 4(a)-(b).

(c) The three kinematic singularities are

- $\theta_3 = 0$: Three coplanar and parallel revolute joint axes.
- $\begin{cases} \theta_2 = 0, \theta_3 = -\frac{\pi}{2}, \theta_4 = -\frac{\pi}{2} \\ \theta_2 = \pi, \theta_3 = \frac{\pi}{2}, \theta_4 = \frac{\pi}{2} \end{cases}$ : Two colinear revolute joint axes.
- $\theta_3 = 0, \theta_4 \neq \pm\frac{\pi}{2}, \theta_5 = 0$ or $\pi$: the case described in Problem 4(b).

(d) From (c), the zero position is clearly singular, so the robot cannot generate wrenches $\mathcal{F}_b$ that are in the null space of $J_b^T$ via the static relation $\tau = J_b^T \mathcal{F}_b$. However, these wrenches are resisted by structural constraints of the mechanism. As an analogy, consider a door attached to the wall via a hinge (or revolute) joint: only forces in the direction of motion of the door can be actively generated, but the door itself can still resist external wrenches in the five-dimensional orthogonal wrench space.

**Problem 5** (30 points)
(a) Substituting the PD control law into the dynamics,

$$\ddot{x} + (2 + k_d)\dot{x} + (1 + k_p)x = 0.$$

Writing the above in the standard second-order form

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0,$$

where $\omega_n$ is the natural frequency and $\zeta$ is the damping ratio. For critical damping $\zeta$ should be 1, or

$$\frac{(2 + k_d)^2}{4(1 + k_p)} = 1.$$

(b) The mass matrix must be symmetric and positive-definite for all values of $\theta$; equivalently, all eigenvalues must be positive (incidentally, all symmetric matrices have real eigenvalues). Also, when the velocity $\dot{\theta}$ is set to zero, the dynamics reduces to $\tau = M(\theta)\ddot{\theta}$. Going through the data measurement pairs,

1. $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$ violates the positive-definiteness condition.

2. $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ violates the symmetricity condition.

3. $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 8 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$ is the only valid measurement pair.

(c) Both $\tilde{M}_1$ and $\tilde{M}_2$ have the same characteristic polynomial $s^2 - 6s + 7$, so at a minimum, the average should also have the same characteristic polynomial as well. Naturally the average, which we'll denote $\bar{M}$, should also be symmetric positive-definite. Putting all these together, we have

$$\bar{M} = \begin{bmatrix} a & b \\ b & c \end{bmatrix},$$

whose characteristic polynomial is $s^2 - (a + b)s + (ac - b^2)$, meaning $a + b = 6$ and $ac - b^2 = 7$. Since there are three variables and only two equations, a degree of freedom remains. If you got this far, you received full credit for this problem, but going further, the orthogonal diagonaliation of $\tilde{M}_1$ and $\tilde{M}_2$ are, after a bit of calculation,

$$\begin{aligned}
\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} &= \begin{bmatrix} \mu_1 & -\nu_1 \\ \nu_1 & \mu_1 \end{bmatrix} \begin{bmatrix} 3 + \sqrt{2} & 0 \\ 0 & 3 - \sqrt{2} \end{bmatrix} \begin{bmatrix} \mu_1 & \nu_1 \\ -\nu_1 & \mu_1 \end{bmatrix} \\
\begin{bmatrix} 3 & \sqrt{2} \\ \sqrt{2} & 3 \end{bmatrix} &= \begin{bmatrix} \mu_2 & -\nu_2 \\ \nu_2 & \mu_2 \end{bmatrix} \begin{bmatrix} 3 + \sqrt{2} & 0 \\ 0 & 3 - \sqrt{2} \end{bmatrix} \begin{bmatrix} \mu_2 & \nu_2 \\ -\nu_2 & \mu_2 \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
\mu_1 &= \frac{1}{\sqrt{4 - \sqrt{2}}}, \quad \nu_1 = \tfrac{\sqrt{2}-1}{\sqrt{4-2\sqrt{2}}} \\
\mu_2 &= \frac{1}{\sqrt{2}}, \qquad \nu_2 = \tfrac{1}{\sqrt{2}}.
\end{aligned}$$

Any real-symmetric matrix $A$ with distinct eigenvalues can be orthogonally diagonalized into the form $A = R^T D R$, where $R$ is orthogonal and $D$ is diagonal. For our problem, $R \in \mathrm{SO}(2)$ with $(\mu, \nu)$

satisfying $\mu^2 + \nu^2 = 1$, or equivalently, $\mu = \cos\phi$ and $\nu = \sin\phi$ for some angle $\phi$. To calculate the average of $\tilde{M}_1$ and $\tilde{M}_2$, therefore, one possibility is to first calculate $\phi_1$ and $\phi_2$ from the relations $\mu_i = \cos\phi_i$ and $\nu_i = \sin\phi_i$, $i = 1, 2$. Once $\phi_1$ and $\phi_2$ are found in this fashion, one can then take the average $\bar{\phi} = (\phi_1 + \phi_2)/2$, and to set $\bar{\mu} = \sin\bar{\phi}$, $\bar{\nu} = \cos\bar{\phi}$, and set

$$\bar{M} = \begin{bmatrix} \bar{\mu} & -\bar{\nu} \\ \bar{\nu} & \bar{\mu} \end{bmatrix} \begin{bmatrix} 3+\sqrt{2} & 0 \\ 0 & 3-\sqrt{2} \end{bmatrix} \begin{bmatrix} \bar{\mu} & \bar{\nu} \\ -\bar{\nu} & \bar{\mu} \end{bmatrix}.$$

**Problem 6** (30 points)
(a) First, observe that $\log PRP^T = P \cdot \log R \cdot P^T$ for any $R, P \in \mathrm{SO}(3)$. (Proof: let $[r] = \log R$, or $e^{[r]} = R$. Then $Pe^{[r]}P^T = PRP^T$ for any $P \in SO(3)$. But $Pe^{[r]}P^T = e^{P[r]P^T}$, so $PRP^T = e^{P[r]P^T}$. Taking the log of both sides, $\log PRP^T = P[r]P^T = P \cdot \log R \cdot P^T$ as claimed.) Then straightforwardly, the shortest path between $R_0' = R_0 P$ and $R_1' = R_1 P$ is $R'(s) = R_0' e^{[r']s}$, where

$$[r'] = \log(R_0'^T R_1') = \log(P^T R_0^T R_1 P) = P^T \log(R_0^T R_1) P$$

using our earlier result. Since $P \in SO(3)$ we can write $r' = P^T r$, from which it follows straightforwardly that $\|r'\| = \|r\|$.
(b) To calculate $R(s)$, first find the shortest path between $R_0$ and $R_1$, which we'll call $R_{01}(s)$; using the previous results,

$$R_{01}(s) = R_0 e^{[r_{01}]s}, \ \ [r_{01}] = \log(R_0^T R_1).$$

Similarly, the shortest path between $R_1$ and $R_2$ is

$$R_{12}(s) = R_1 e^{[r_{12}]s}, \ \ [r_{12}] = \log(R_1^T R_2).$$

$R(s)$ is then found from shortest path between $R_{01}(s)$ and $R_{12}(s)$ as follows:

$$R(s) = R_{01}(s) e^{[r]s}, \ \ [r] = \log(R_{01}(s)^T R_{12}(s)).$$