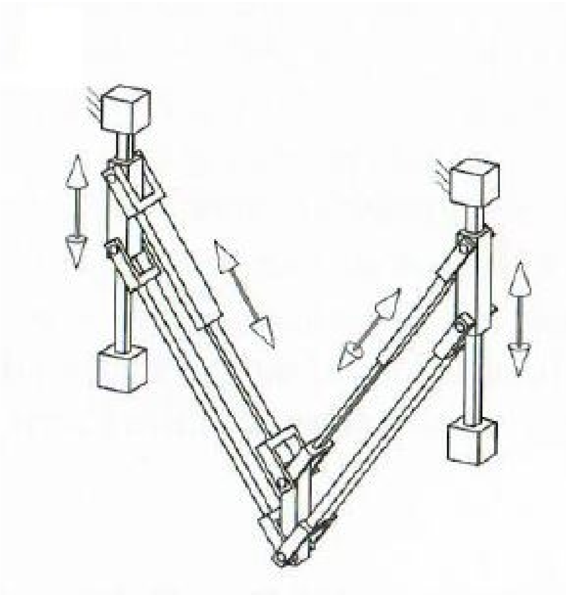
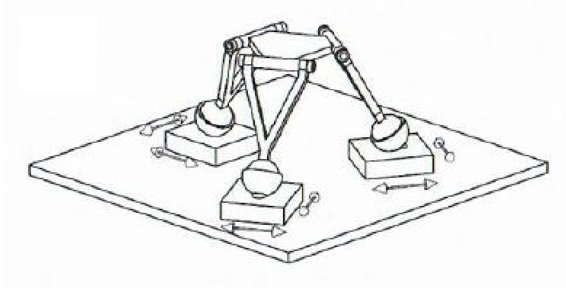
M2794.002700 Introduction to Robotics
Midterm Examination 1
April 6, 2017
CLOSED BOOK, CLOSED NOTES

Problem 1

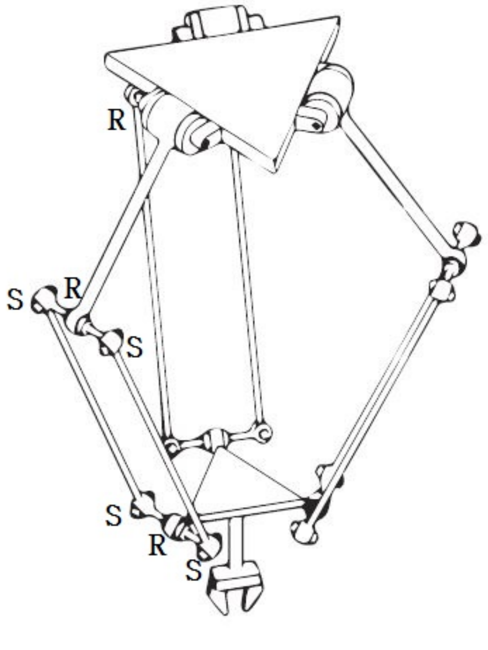
(a) Use Grübler's formula to find the degrees of freedom of the mechanism shown in Figure 1-(a).
(b) Use Grübler's formula to find the degrees of freedom of the 3-PPSR mechanism of Figure 1-(b) (note that the three boxes can only slide in the direction of the arrows; they cannot rotate).
(c) The Delta robot of Figure 1-(c) consists of two platforms—the lower one is mobile, the upper one is fixed to the ceiling—connected by three leg identical legs: each leg is an RR serial chain connected to a close-loop parallelogram linkage (each of the joint types are labelled in the figure). This Delta robot has 3 degrees of freedom. Use Grübler's formula to find the degrees of freedom of this robot. Is your answer 3? If not, explain why Grübler's formula fails.
(d) Figure 1-(d) shows another version of the Delta robot in which the S joints are now replaced by R joints. Use Grübler's formula to find the degrees of freedom of this robot. Can this robot move? Explain your answer, including all assumptions made.



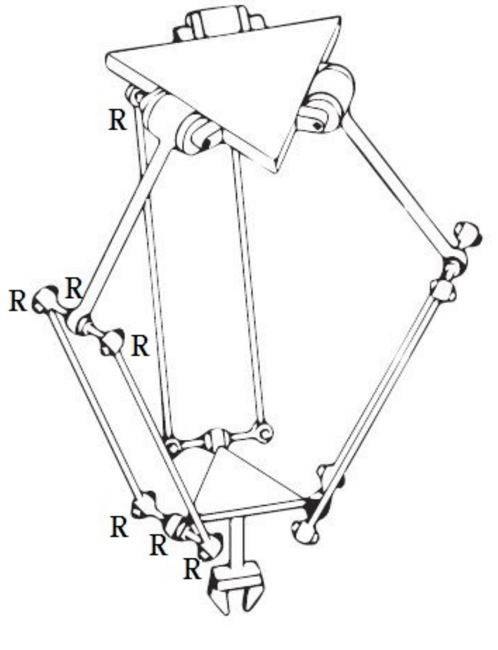
(a) Mechanism for Problem 1-(a).



(b) 3-PPSR mechanism for Problem 1-(b).



(c) Delta robot ver.1 for Problem 1-(c).



(d) Delta robot ver.2 for Problem 1-(d).

Figure 1: Various mechanisms for Problem 1.

Problem 2

The planar rigid object shown in Figure 2 has corners A, B, $\cdots$, F, and O. The fixed frame is attached at O as shown, and each grid is of size 1×1. Points P_1, P_2, P_3, and P_4 are point contacts (P_1 and P_3 lie at the center of $\overline{BC}$ and $\overline{QE}$, respectively).

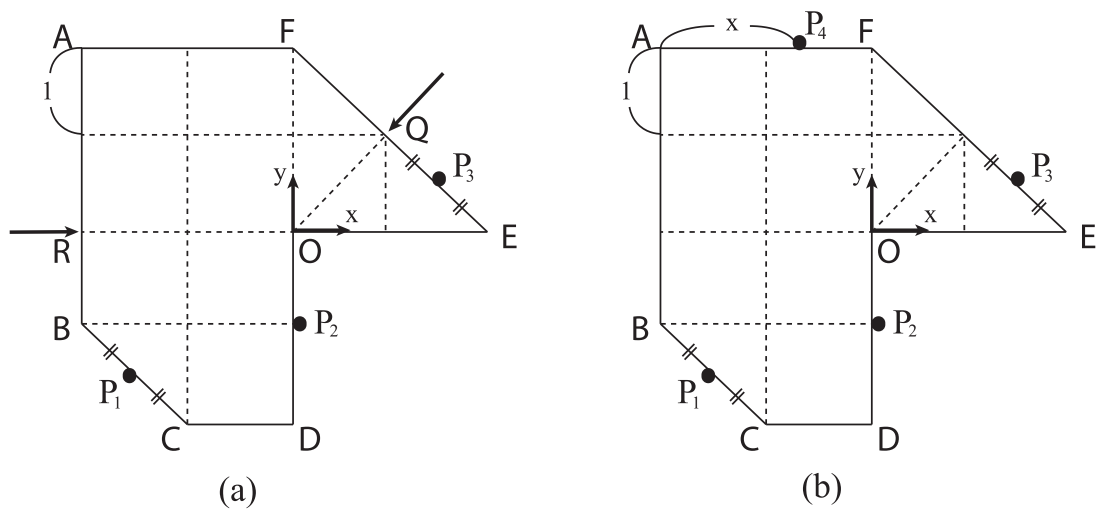


Figure 2: Planar rigid object for Problem 2.

(a) In Figure 2-(a), the point contacts P_1, P_2, and P_3 are frictionless. (For this problem, you ignore the external forces drawn at R and Q.) Is this grasp force closure? Explain your answer.

(b) Now assume that arbitrary external normal forces R and Q are being applied to the object as shown in Figure 2-(a). Point contacts P_1, P_2, and P_3 are frictionless as before. Are these three point contacts sufficient to resist the external forces R and Q?

(c) In Figure 2-(b), assume that point contacts P_1, P_2, P_3, and P_4 are frictionless. Determine the range of x that makes the grasp force closure.

Problem 3

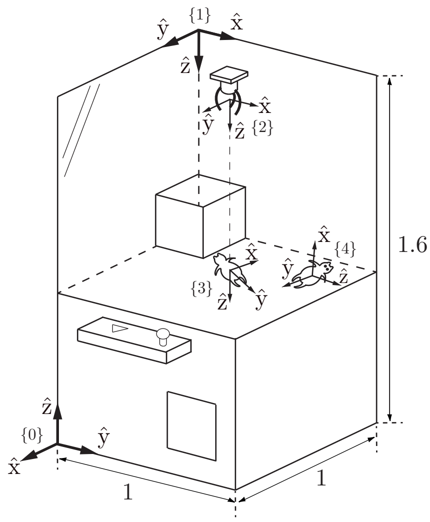


Figure 3: Game arcade claw machine for Problem 3.

Figure 3 shows a game arcade claw machine (only two dolls are remaining). Reference frames {0}, {1}, {2}, {3},and {4} are attached to the base, ceiling, the claw, and each doll as shown. The following vectors and matrices are defined:

- $p_{ij} \in \mathbb{R}^3$ is the vector from frame $\{i\}$ to frame $\{j\}$, expressed in frame $\{i\}$ coordinates.
- $R_{ij} \in SO(3)$ is the 3×3 rotation matrix describing the orientation of frame $\{j\}$ as seen from frame $\{i\}$.
- $T_{ij} \in SE(3)$ is the 4×4 rigid body transformation matrix describing the position and orientation of frame $\{j\}$ as seen from frame $\{i\}$.

In what follows, T_{03} is given as

$$T_{03} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -0.6 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0.5 \\ 0 & 0 & -1 & 0.7 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

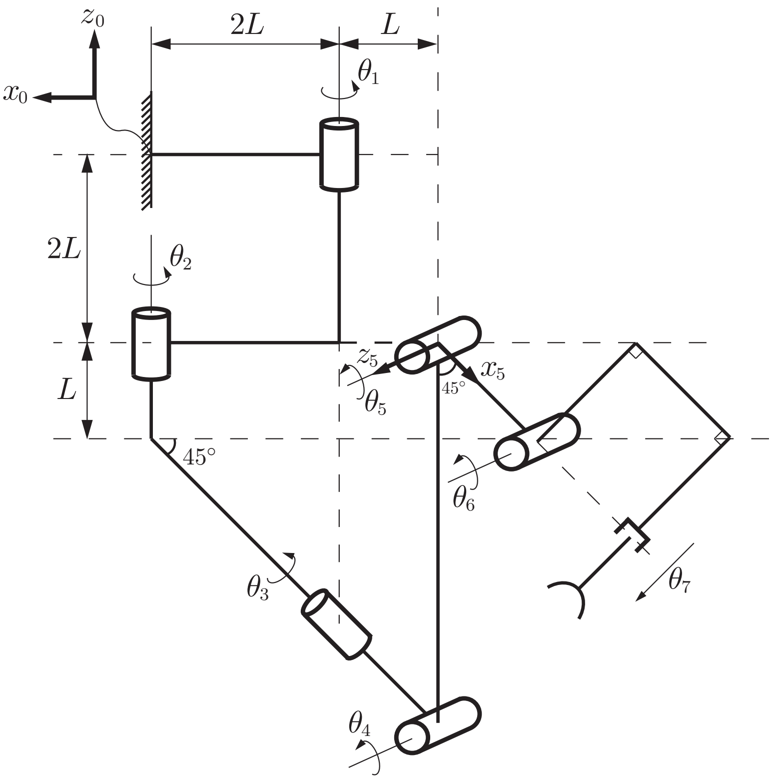
(a) Find p_{13}.
(b) Suppose $R_{34} = e^{[\hat{\omega}]\theta}$, where $\hat{\omega} \in \mathbb{R}^3$ is a unit vector in the direction of $(-2, 5, 1)^T$ expressed in frame $\{3\}$, and $\theta = \frac{\pi}{2}$. Write down an explicit expression for R_{34}.

(c) Now you are trying to pick up the doll at frame $\{3\}$. At the instant the button is pressed, the claw moves downward in the z-direction while also rotating about the z-axis and swinging about y-axis. The claw's movement can be described as the following matrix in terms of time t:

$$T_{12}(t) = \begin{bmatrix} & \mathrm{Rot}(\hat{z}, wt) \cdot \mathrm{Rot}(\hat{y}, \frac{\pi}{6}\sin t) & & p_{13,x} \\ & & & p_{13,y} \\ & & & vt + 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $w = \frac{1}{8}$(rad/s) and $v = \dfrac{0.4}{\pi}$(m/s). Suppose the claw stops when the origins of frames $\{2\}$ and $\{3\}$ meet. Find T_{23} at the instant the claw stops.

Problem 4
The RRRRRRP spatial open chain of Figure 4-(a) is shown in its zero position. Frames {0}, {5}, and some Denavit-Hartenberg parameters are given. Attach appropriate link frames and find the remaining Denavit-Hartenberg parameters.



(a)

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1				
2			$-3L$	
3	45°			
4				
5		$4L$		
6				θ_6
7				0

(b)

Figure 4: RRRRRRP open chain

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Problem 1 (50 Points)
(a) The movement of this mechanism is constrained to the plane. Applying Grübler's formula for planar mechanism yields:
$N = 9$ (links) $+1$ (ground) $= 10$
$J = 8$ (R joints) $+ 4$ (P joints)
$\Sigma f_i = 12$
$\text{dof} = 3(N - 1 - J) + \Sigma f_i = 3(10 - 1 - 12) + 12 = 3$

(b) Each box can be regarded as a link connected to ground by a 2-dof PP joint. Applying Grübler's formula for spatial mechanism yields:
$N = 7$ (links) $+1$ (ground) $= 8$
$J = 3$ (R joints) $+ 3$ (S joints) $+ 3$ (2-dof joints) $= 9$
$\Sigma f_i = 3 \times 1$ (R joints) $+3 \times 3$ (S joints) $+2 \times 3$ (2-dof joints) $= 18$
$\text{dof} = 6(N - 1 - J) + \Sigma f_i = 6(8 - 1 - 9) + 18 = 6$

(c) Applying the spatial version of Grübler's formula leads to the following:
$N = 16$ (links) $+1$ (ground) $= 17$
$J = 9$ (R joints) $+ 12$ (S joints) $= 21$
$\Sigma f_i = 9 \times 1$ (R joints) $+3 \times 12$ (S joints) $= 45$
$\text{dof} = 6(N - 1 - J) + \Sigma f_i = 6(17 - 1 - 21) + 45 = 15$

The twelve additional dof obtained from Grubler's formula can be attributed to the torsional rotation of the links (links 1,2,3,4) about their respective axes, which has no effect on the movement of the robot's moving platform.

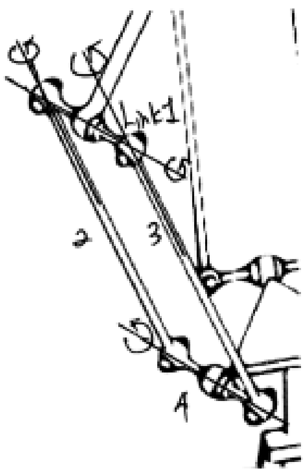


Figure 1: rotation of the links (link 1,2,3,4)

(d) Applying the spatial version of Grübler's formula leads to the following.
$N = 16$ (links) $+1$ (ground) $= 17$
$J = 9$ (R joints) $+$ 12 (R joints) $= 21$
$\Sigma f_i = 21 \times 1$ (R joints) $= 21$
$\text{dof} = 6(N - 1 - J) + \Sigma f_i = 6(17 - 1 - 21) + 21 = -9$

Grubler's formula would thus seem to imply that the mechanism is overconstrained. However, each parallelogram linkage has one degree of freedom of motion, so that each leg of the Delta robot is kinematically equivalent to an RUU chain (see *Modern Robotics*, p. 22). Replacing each parallelogram linkage by an RUU chain and applying the spatial version of Grübler's formula:
$N = 7$ (links) $+1$ (ground) $= 8$
$J = 3$ (R joints) $+$ 6 (U joints) $= 9$
$\Sigma f_i = 3 \times 1$ (R joints) $+6 \times 2$ (U joints) $= 15$
$\text{dof} = 6(N - 1 - J) + \Sigma f_i = 6(8 - 1 - 9) + 15 = 3$

Problem 2 (50 Points)

(a) Planar force closure requires a minimum of four frictionless point contacts (provided that none are placed at corners). Geometrically, each frictionless point contact maps to the vertex of a tetrahedron that must enclose an open ball centered at the origin. For the problem as stated, only three frictionless point contacts are given, making force closure impossible to achieve.

(b) The sum of all the forces and moments exerted on the block must be zero to keep it stationary:

$$\sum_i f_i = 0 \text{ and } \quad \sum_i \tau_i = 0.$$

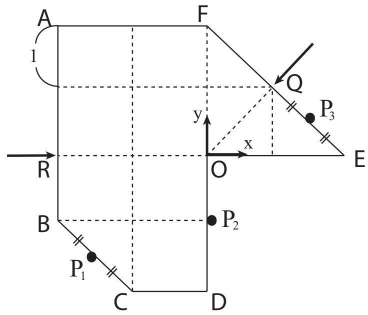


Figure 2: Three point contacts to resist the external forces.

Let f_i be the force at each point contact P_i, and F_R, F_Q be the forces exerted at points R and Q, respectively. The problem can then be reformulated as follows:

$$\text{There exists} \quad a_i \geq 0 (i = 1, 2, 3) \text{ such that } \sum_{i=1}^{3} a_i f_i + b_R F_R + b_Q F_Q, \text{ for all } b_R, b_Q \geq 0 \quad (1)$$

$$\vec{\tau}_2 + \vec{\tau}_3 = 0 \quad (2)$$

where $\vec{\tau}_i$ is the moment generated by f_i about O and a_i, b_R, b_Q are the linear coefficients of the corresponding forces. Note that f_1, F_R and F_Q do not exert any moments on the block since their moment arms are all zero. If there exists any non-negative solution (a_i, b_R, b_Q) to (1)-(2), the block can be kept stationary. However, both a_2 and a_3 must be zero to satisfy (2) since both of their torques are directed in the negative z-direction. Since $a_1 f_1 + b_R F_R + b_Q F_Q = 0$ can't be true for all $b_R, b_Q \geq 0 \in \mathbb{R}$, this grasp cannot resist external forces, that is, the given point contacts are not enough.

(c) It is possible to translate the forces along their lines of action as shown in Figure 3, because translations do not affect resultant forces or moments exerted on the block (think about the moment arm of forces.) Translating the forces along their lines of action and applying Nguyen's theorem, it can be determined that the position of contact P_4 must satisfy $0 < x < 1$.

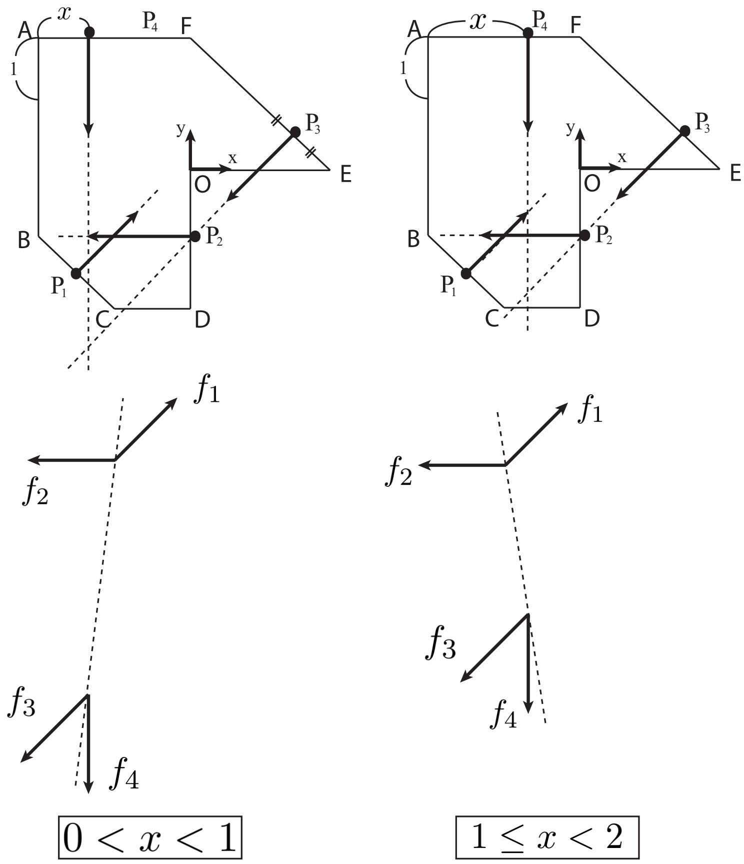


Figure 3: Two configuration of the forces

Problem 3 (50 points)

(a) From the relation $p_{13} = p_{10} + R_{10}p_{03}$,

$$p_{13} = \begin{bmatrix} 0 \\ 1 \\ 1.6 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -0.6 \\ 0.5 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.4 \\ 0.9 \end{bmatrix}.$$

(b) Since $\hat{\omega}$ is a unit vector, $\hat{\omega} = \frac{1}{\sqrt{30}}(-2, 5, 1)^T$. From the Rodrigues formula,

$$\begin{aligned} R_{34} &= I + \sin\theta[\hat{\omega}] + (1 - \cos\theta)[\hat{\omega}]^2 \\ &= I + \frac{1}{\sqrt{30}} \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix} + \frac{1}{30} \begin{bmatrix} -26 & -10 & -2 \\ -10 & -5 & 5 \\ -2 & 5 & -29 \end{bmatrix} \\ &= \frac{1}{\sqrt{30}} \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & 2 \\ -5 & -2 & 0 \end{bmatrix} + \frac{1}{30} \begin{bmatrix} 4 & -10 & -2 \\ -10 & 25 & 5 \\ -2 & 5 & 1 \end{bmatrix}. \end{aligned}$$

(c) From the relation $vt + 0.1 = p_{13,z} = 0.9$, it takes 2π seconds for the claw to come to a stop. At time $t = 2\pi$, T_{12} becomes

$$T_{12} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 0.5 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0.4 \\ 0 & 0 & 1 & 0.9 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the claw stops when the origins of frames $\{2\}$ and $\{3\}$ meet, $p_{23} = (0, 0, 0)^T$; therefore we only need to consider the rotation matrices. From the relation $R_{23} = R_{20}R_{03} = (R_{01}R_{12})^T R_{03}$,

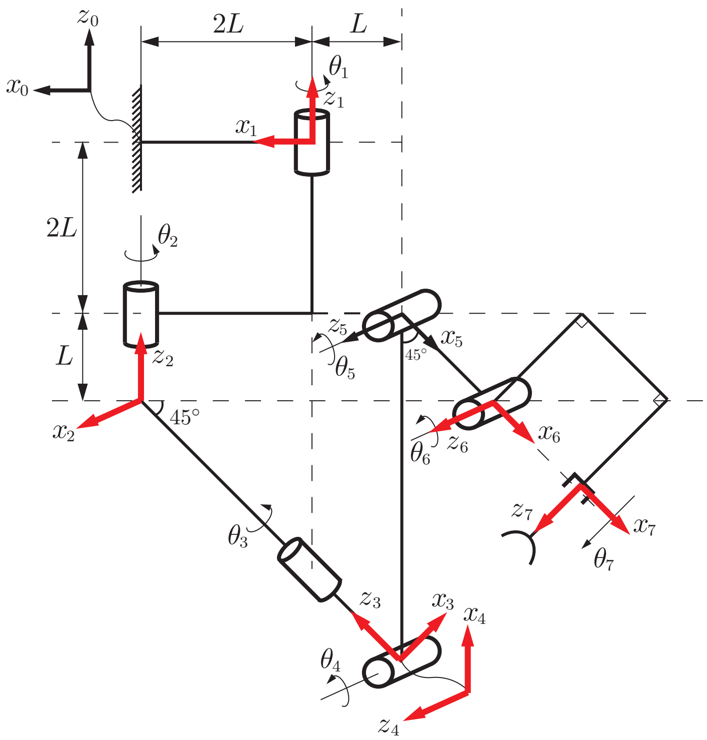
$$R_{01}R_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$R_{23} = (R_{01}R_{12})^T R_{03} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}-\sqrt{6}}{4} & \frac{\sqrt{2}+\sqrt{6}}{4} & 0 \\ -\frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

T_{23} therefore becomes

$$T_{23} = \begin{bmatrix} \frac{\sqrt{2}-\sqrt{6}}{4} & \frac{\sqrt{2}+\sqrt{6}}{4} & 0 & 0 \\ -\frac{\sqrt{2}+\sqrt{6}}{4} & \frac{\sqrt{2}-\sqrt{6}}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 4 (50 Points)
One possible set of link frames is shown in Figure 4-(a). The corresponding Denavit-Hartenberg parameters are shown in Figure 4-(b).



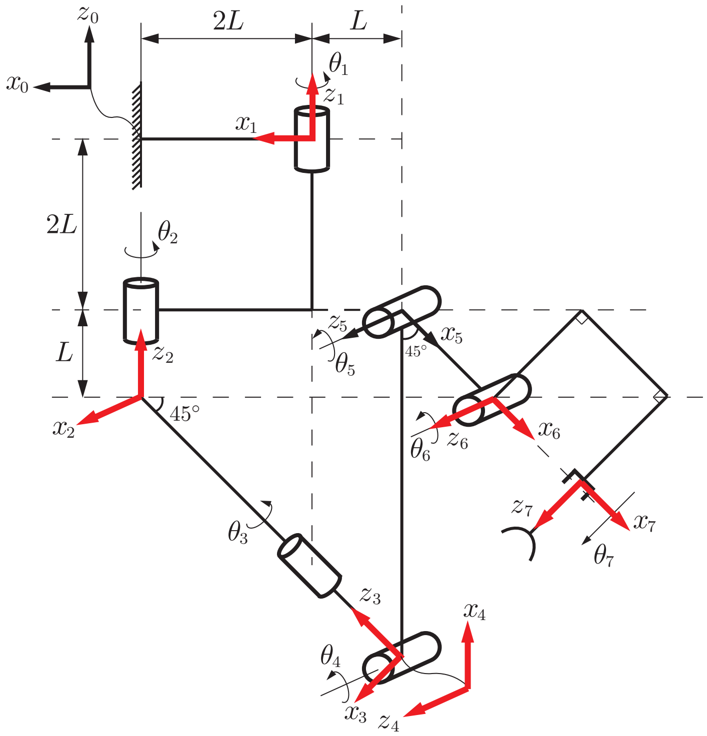
(a)

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	$-2L$	0	θ_1
2	0	$2L$	$-3L$	$90°+\theta_2$
3	$45°$	0	$-3\sqrt{2}L$	$90°+\theta_3$
4	$90°$	0	0	$45°+\theta_4$
5	0	$4L$	0	$-135°+\theta_5$
6	0	$\sqrt{2}L$	0	θ_6
7	$90°$	$\sqrt{2}L$	θ_7	0

(b)

Figure 4: One possible set of link frames and its corresponding Denavit-Hartenberg parameters

Another possible set of link frames is shown in Figure 5-(a). The corresponding Denavit-Hartenberg parameters are shown in Figure 5-(b). Other solutions may exist.



(a)

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	$-2L$	0	θ_1
2	0	$2L$	$-3L$	$90^\circ+\theta_2$
3	45°	0	$-3\sqrt{2}L$	$-90^\circ+\theta_3$
4	-90°	0	0	$-135^\circ+\theta_4$
5	0	$4L$	0	$-135^\circ+\theta_5$
6	0	$\sqrt{2}L$	0	θ_6
7	90°	$\sqrt{2}L$	θ_7	0

(b)

Figure 5: Another possible set of link frames and its corresponding Denavit-Hartenberg parameters

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Exam II
May 11, 2017
CLOSED BOOK, CLOSED NOTES

Problem 1

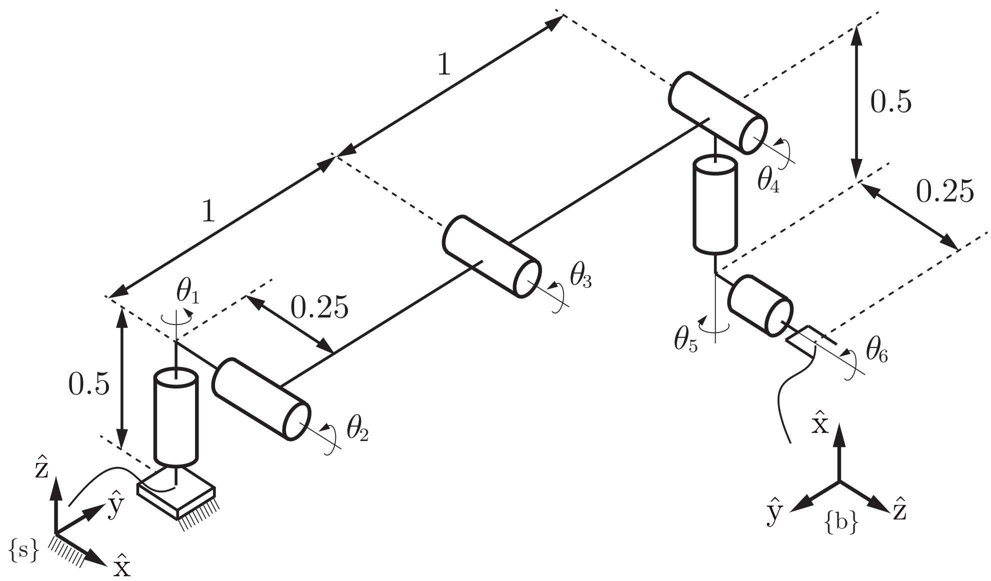


Figure 1: UR5 6R robot arm in its zero position.

Figure 1 shows a UR5 6R robot arm in its zero position, with space and end-effector frames chosen as shown.

(a) Suppose $\theta_5 = \theta_6 = 0$. For the desired end-effector position and orientation

$$T_{sb} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ -1 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & -1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

find all inverse kinematics solution $(\theta_1, \theta_2, \theta_3, \theta_4)$. How many inverse kinematics solutions can you find?

(b) Show that $[\mathrm{Ad}_T]$ for any $T \in SE(3)$ is always nonsingular. Use this fact to argue that the space Jacobian and body Jacobian always have the same rank.

(c) Consider the following minimization problem that arises in computer vision:

$$\min L(\omega, v) = \|b(v) - A(v)\omega\|^2,$$

where $\omega, v \in \mathbb{R}^3$, $b(v) \in \mathbb{R}^{3n}$, $A(v) \in \mathbb{R}^{3n\times 3}$, and $v \in \mathbb{R}^3$ must satisfy $v^T v = 1$. Find an expression for the optimal ω as a function of v.

Problem 2

The spatial RRRRRP open chain of Figure 2 is shown in its zero position, with space and end-effector frames chosen as shown.

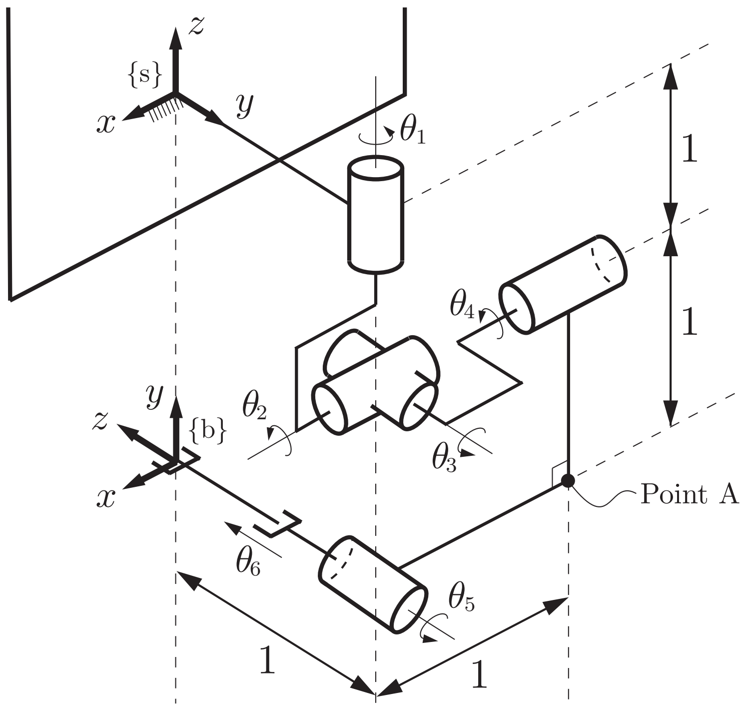


Figure 2: RRRRRP open chain

(a) Derive its forward kinematics in the form

$$T_{sb} = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M e^{[S_4]\theta_4} e^{[S_5]\theta_5} e^{[S_6]\theta_6},$$

where $M \in SE(3)$.

(b) Is the zero position a kinematic singularity? Explain your answer.

(c) At the zero position, let $\dot{\theta} = (1, 1, 1, 1, 1, 1)^T$. Find the linear velocity of the end-effector in {s} frame coordinates.

(d) At the zero position, two external forces, f_{ext_1} and f_{ext_2}, are applied to the open chain. $f_{ext_1} = (-f_1, -f_2, -f_3)^T$ is applied to the end-effector, while $f_{ext_2} = (-f_4, -f_5, -f_6)^T$ is applied to point A shown in the figure. Both are expressed in {s} frame coordinates. Define $f = (f_1, f_2, f_3, f_4, f_5, f_6)^T$, and let $\tau = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6)^T$ be the input joint torques required to maintain static equilibrium. Then τ and f satisfy the following equation:

$$\tau = K f.$$

Find matrix K.

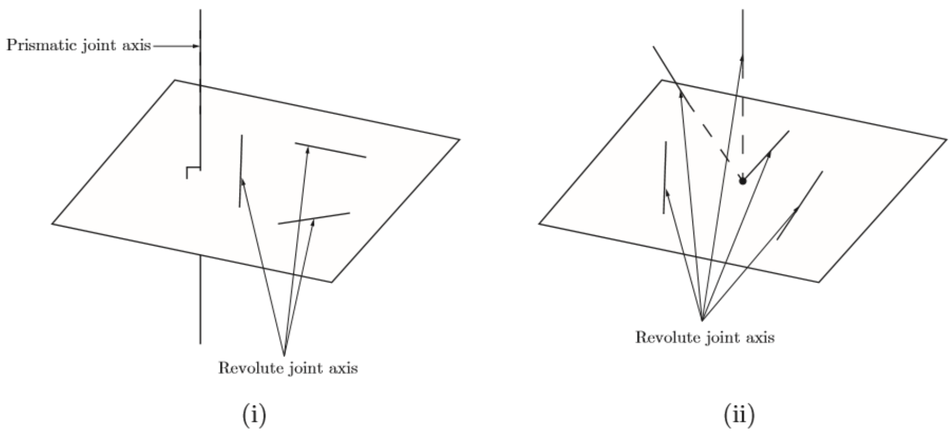
(e) Suppose the input joint torques are $\tau = (7, 3, 7, 3, 0, 4)^T$. Find the minimum norm f that satisfies the static equilibrium condition.

Problem 3

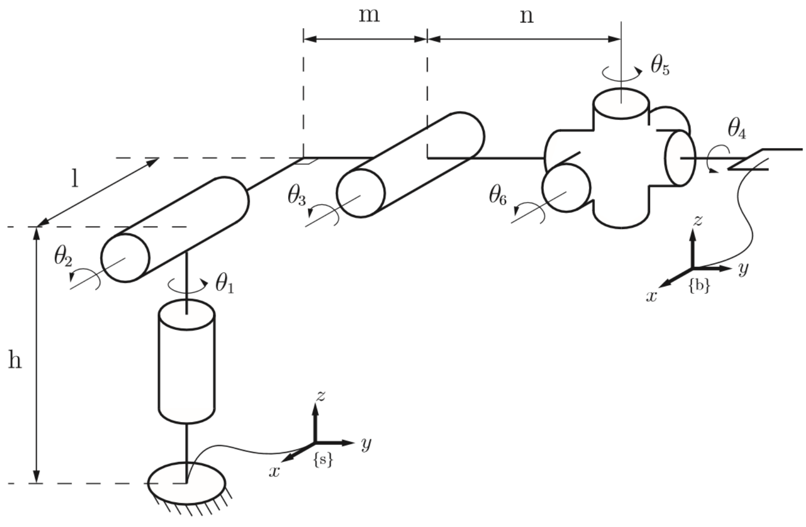
(a) A six-dof spatial open chain has three of its revolute joint axes coplanar, and a prismatic joint axis normal to the plane spanned by the three coplanar revolute joint axes (see Figure 3(a)-(i)). Is this configuration a singularity? Explain your answer.

(b) A six-dof spatial open chain has three of its revolute joint axes intersecting at a common point; this comon point lies on the plane spanned by two other revolute joint axes (see Figure 3(a)-(ii)). Is this configuration a singularity? Explain your answer.

(c) Try to find as many singularities of the 6R PUMA-type arm shown in Figure 3(b). For each singularity, explain the type using the screw conditions.



(a) Kinematic singularity involving prismatic and revolute joints.



(b) 6R PUMA-type arm.

Figure 3: Figures for Problem 3.

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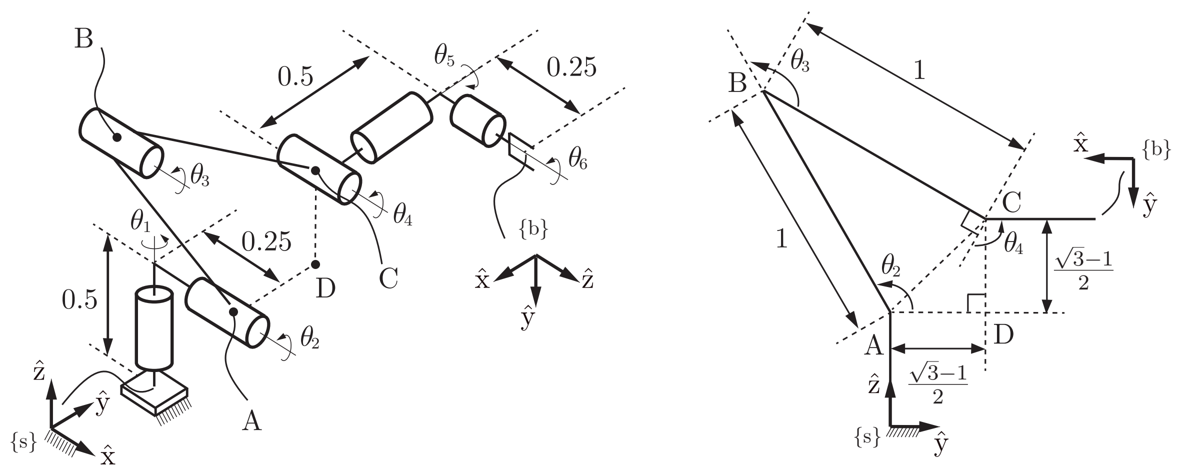
Problem 1 (50 Points)

(a) The desired end-effector position and orientation is represented as

$$T_{sb} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} \\ -1 & 0 & 0 & \frac{\sqrt{3}}{2} \\ 0 & -1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\theta_5 = 0$, it is only θ_1 that can affect the z-axis orientation of the end-effector. The desired z-axis orientation of the end-effector is $(1, 0, 0)^T$, which is the same as that of zero position. θ_1 must therefore be zero.



(a) UR5 in the desired position and orientation

(b) Projected view (elbow-up)

Figure 1: UR5 6R robot arm for Problem 1

As shown in Figure 1, since $\theta_6 = 0$, axis of joint 5 must be parallel to y-axis of space frame {s} to yield the desired x-axis and y-axis orientation of the end-effector. From the desired position of the end-effector and given link lengths, we can derive the projected view as Figure 1-(b). We can easily find out $\triangle ADC$ is an isosceles right triangle. Hence, we can obtain $\angle ACD, \angle CAD$, and $\overline{AC}$ as follows:

$$\angle CAD = \angle ACD = \frac{1}{4}\pi,$$
$$\overline{AC} = \sqrt{2} \cdot \frac{\sqrt{3}-1}{2} = \frac{\sqrt{6}-\sqrt{2}}{2}.$$

Then, from the law of cosines and the property of an isosceles triangle, $\angle ABC$, $\angle BAC$, and $\angle BCA$ can be determined:

$$\angle ABC = \cos^{-1}\left(\frac{1^2+1^2-(\frac{\sqrt{6}-\sqrt{2}}{2})^2}{2\cdot 1\cdot 1}\right) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{1}{6}\pi,$$
$$\angle BAC = \angle BCA = \frac{1}{2}(\pi - \angle ABC) = \frac{5}{12}\pi.$$

For elbow-up case as shown in Figure 1-(b),

$$\theta_2 = \angle CAD + \angle BAC = \frac{2}{3}\pi,$$
$$\theta_3 = -\pi + \angle ABC = -\frac{5}{6}\pi,$$
$$\theta_4 = \angle BCA + \angle ACD = \frac{2}{3}\pi.$$

For elbow-down case,

$$\theta_2 = \angle CAD - \angle BAC = -\frac{1}{6}\pi,$$
$$\theta_3 = \pi - \angle ABC = \frac{5}{6}\pi,$$
$$\theta_4 = -\angle BCA + \angle ACD = -\frac{1}{6}\pi.$$

We therefore have two inverse kinematics solutions:

$$(\theta_1,\theta_2,\theta_3,\theta_4) = (0, \frac{2}{3}\pi, -\frac{5}{6}\pi, \frac{2}{3}\pi),$$
$$(\theta_1,\theta_2,\theta_3,\theta_4) = (0, -\frac{1}{6}\pi, \frac{5}{6}\pi, -\frac{1}{6}\pi).$$

(b)

(i) Let $T \in SE(3)$ be

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad [Ad_T] = \begin{bmatrix} R & 0 \\ [\mathrm{p}]R & R \end{bmatrix}$$

For any $R \in SO(3)$ and $p \in \mathbb{R}^3$, $[Ad_T]$ always has an inverse.

$$[Ad_T]^{-1} = \begin{bmatrix} R^T & 0 \\ -[R^Tp]R^T & R^T \end{bmatrix}$$

Since $[Ad_T]$ is invertible, it is nonsingular.

(ii) $J_s(\theta) = [Ad_T]J_b(\theta)$.
$\Rightarrow rank(J_s(\theta)) = rank([Ad_T]J_b(\theta)) = rank(J_b(\theta))$.
Reason : For $A \in \mathbb{R}^{n\times m}, P \in \mathbb{R}^{n\times n}$, if P is nonsingular, $rank(A) = rank(PA)$

(c) For a given v, this is an unconstrained minimization problem for a function $L(w)$

$$\min_{w\in\mathbb{R}^3} \|b(v) - A(v)w\|^2$$

$$\text{FONC} : \tfrac{\partial L}{\partial w}(w^*) = 0$$
$$\tfrac{\partial L}{\partial w}(w^*) = -2b^T A + 2w^{*T} A^T A = 0$$
$$\Rightarrow A^T A w^* = A^T b$$
$$\Rightarrow \text{If } A^T A \text{ is nonsingular, } w^* = (A^T A)^{-1} A^T b.$$

Problem 2 (80 Points)

(a) Tweak the standard POE formula using transformation of twist as follows,

$$\begin{aligned} T_{sb} &= e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} e^{[S_4]\theta_4} e^{[S_5]\theta_5} e^{[S_6]\theta_6} M \\ &= e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M e^{[S'_4]\theta_4} e^{[S'_5]\theta_5} e^{[S'_6]\theta_6}, \\ \text{where} \quad [S_i] &= \begin{bmatrix} [w_i] & v_i \\ 0 & 0 \end{bmatrix}, \quad [S'_i] = [Ad_{M^{-1}}(S_i)] = M^{-1}[S_i]M. \end{aligned}$$

Geometrical interpretation of S'_i is simply changing the coordinates from $\{s\}$ frame to {b} frame. At zero-position, the end-effector is located at $p = (0, 0, -2)^T$ and its orientation is

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus, rigid body transformation matrix of the end-effector at zero-position is

$$M = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To compute screw $S_i = (w_i, v_i)^T$ of each revolute joint, get w_i as its joint axis and set $v_i = -q_i \times w_i$ where q_i is an arbitrary point on its joint axis.

i	w_i	q_i	v_i	i	w'_i	q'_i	v'_i
1	$(0, 0, 1)$	$(0, 1, 0)$	$(1, 0, 0)$	4	$(1, 0, 0)$	$(0, 1, -1)$	$(0, -1, -1)$
2	$(1, 0, 0)$	$(0, 1, -1)$	$(0, -1, -1)$	5	$(0, 0, -1)$	$(0, 0, 0)$	$(0, 0, 0)$
3	$(0, 1, 0)$	$(0, 0, -1)$	$(1, 0, 0)$	6	$(0, 0, 0)$	$\times$	$(0, 0, 1)$

(b) Kinematic singularity occurs when there exists:

1. Two colinear revolute joints
2. Three coplanar and parallel revolute joints
3. Four revolute joints intersecting at a common point
4. Four coplanar revolute joints
5. Six revolute joints intersecting a common line

The manipulator of the problem falls under the first and the third condition: the axes 2 and 4 are colinear revolute joints and the axes 1, 2, 3, and 4 are intersecting at a common point. Thus, the manipulator at zero position is obviously kinematic singularity.

On the other hand, kinematic singularity can be also determined by rank of Jacobian matrix. If any Jacobian matrix expressed in any frame is not full rank, then it loses at least one degree of freedom, that is, kinematic singularity. From (a), it is straightfoward to show that the space Jacobian has rank of 5.

(c) There are mainly three ways to compute the linear velocity of the end-effector:

1. Compute the body Jacobian J_b and $V_b = J_b\dot{\theta}$. Then, pre-multiply R_{sb} to express it in {s} frame: $v = R_{sb}v_b$.
2. Use the transformation of velocity: $V_b = [Ad_{T_{bs}}]V_s = [Ad_{T_{bs}}]J_s\dot{\theta}$. Then, pre-multiply R_{sb} to express it in {s} frame: $v = R_{sb}v_b$.
3. By the definition of spatial linear velocity $v_s = \dot{p} - w_s \times p$, the end-effector velocity expressed in {s} frame is $v = \dot{p} = v_s + w_s \times p$.

Using the third method, space Jacobian at zero position is:

$$J_s = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 \end{bmatrix}.$$

Therfore,

$$V_s = J_s\dot{\theta} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 4 \\ -3 \\ -2 \end{bmatrix}.$$

Thus, the spatial linear velocity $v_s = (4, -3, -2)^T$, angular velocity $w_s = (2, 2, 1)^T$, and $p = p_{sb} = (0, 0, -2)^T$. Then, it is straightfoward to compute the velocity of the end-effector $\dot{p}$ as follows,

$$\dot{p} = v_s + w_s \times p = \begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}.$$

(d) Since static equilibrium is maintained, the forces that the robot is generating are $-f_{ext_1} = (f_1, f_2, f_3)^T$ and $-f_{ext_2} = (f_4, f_5, f_6)^T$ respectively. Vectors from the origin of frame {s} to the origin of frame {b} and to point A expressed in {s} frame coordinates are

$$r_s = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \; r_a = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$

Therefore, the spatial forces that the robot is generating are

$$\mathcal{F}_s = \begin{bmatrix} r_s \times (-f_{ext_1}) \\ -f_{ext_1} \end{bmatrix} = \begin{bmatrix} 2f_2 \\ -2f_1 \\ 0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}, \; \mathcal{F}_a = \begin{bmatrix} r_a \times (-f_{ext_2}) \\ -f_{ext_2} \end{bmatrix} = \begin{bmatrix} 2f_5 + f_6 \\ -2f_4 + f_6 \\ -f_4 - f_5 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix}.$$

Denote i-th column of J_s as J_i $(i = 1, 2, 3, 4, 5, 6)$. Joint torques τ_5 and τ_6 are affected only by $\mathcal{F}_s$:

$$\begin{aligned}\tau_5 &= J_5^T \mathcal{F}_s = 0,\\ \tau_6 &= J_6^T \mathcal{F}_s = -f_2.\end{aligned}$$

Joint torques τ_1, τ_2, τ_3 and τ_4 are affected by $\mathcal{F}_s$ and $\mathcal{F}_a$:

$$\begin{aligned}\tau_1 &= J_1^T(\mathcal{F}_s + \mathcal{F}_a) = f_1 - f_5,\\ \tau_2 &= J_2^T(\mathcal{F}_s + \mathcal{F}_a) = f_2 - f_3 + f_5,\\ \tau_3 &= J_3^T(\mathcal{F}_s + \mathcal{F}_a) = -f_1 - f_4 + f_6,\\ \tau_4 &= J_4^T(\mathcal{F}_s + \mathcal{F}_a) = f_2 - f_3 + f_5.\end{aligned}$$

Putting together,

$$\tau = \begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \tau_4\\ \tau_5\\ \tau_6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & -1 & 0 & 1 & 0\\ -1 & 0 & 0 & -1 & 0 & 1\\ 0 & 1 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}f_1\\ f_2\\ f_3\\ f_4\\ f_5\\ f_6\end{bmatrix} = Kf.$$

Therefore,

$$K = \begin{bmatrix}1 & 0 & 0 & 0 & -1 & 0\\ 0 & 1 & -1 & 0 & 1 & 0\\ -1 & 0 & 0 & -1 & 0 & 1\\ 0 & 1 & -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\end{bmatrix}.$$

(e) Substituting the input joint torques $\tau = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6)^T = (7, 3, 7, 3, 0, 4)^T$ to the static equilibrium condition,

$$\begin{aligned}7 &= f_1 - f_5,\\ 3 &= f_2 - f_3 + f_5,\\ 7 &= -f_1 - f_4 + f_6,\\ 3 &= f_2 - f_3 + f_5,\\ 0 &= 0,\\ 4 &= -f_2.\end{aligned}$$

The equations can be simplified as

$$\begin{aligned}f_2 &= -4,\\ f_1 - f_5 &= 7,\\ -f_1 - f_4 + f_6 &= 7,\\ -f_3 + f_5 &= 7.\end{aligned}$$

Let $x = (f_1, f_3, f_4, f_5, f_6)^T$ and $b = (7, 7, 7)^T$. Then,

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} = Ax = b.$$

Since f_2 is constant, the optimization problem can be formulated as:

$$\begin{aligned} &\text{minimize} \quad f(x) = \tfrac{1}{2} x^T x \\ &\text{subject to } g(x) = Ax - b = 0. \end{aligned}$$

From the first-order necessary condition for optimality,

$$\frac{\partial f(x)}{\partial x} + \lambda^T \frac{\partial g(x)}{\partial x} = x^T + \lambda^T A = 0,$$

where $\lambda \in \mathbb{R}^3$. Then,

$$x = -A^T \lambda.$$

Multiplying A to both sides,

$$Ax = -AA^T \lambda.$$

Since rank of A is 3, rank of $AA^T \in \mathbb{R}^{3\times 3}$ is also 3 and therefore invertible. Thus,

$$\lambda = -(AA^T)^{-1} Ax.$$

From the equality constraint $g(x) = 0$, $Ax = b$. Substituting to the equation above,

$$\lambda = -(AA^T)^{-1} b.$$

Therefore, the x that minimizes the objective function is

$$\begin{aligned} x &= -A^T \lambda \\ &= A^T (AA^T)^{-1} b \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 7 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ -9 \\ -6 \\ -2 \\ 6 \end{bmatrix}. \end{aligned}$$

Therefore, the minimum norm f that satisfies the static equilibrium condition is

$$f = \begin{bmatrix} 5 \\ -4 \\ -9 \\ -6 \\ -2 \\ 6 \end{bmatrix}.$$

Problem 3 (50 Points)

(a) Suppose that the the prismatic joint axis is coincident with the z-axis of the fixed frame and the three revolute joint axes are on the xy-plane of the fixed frame as shown in Figure 2-(i).

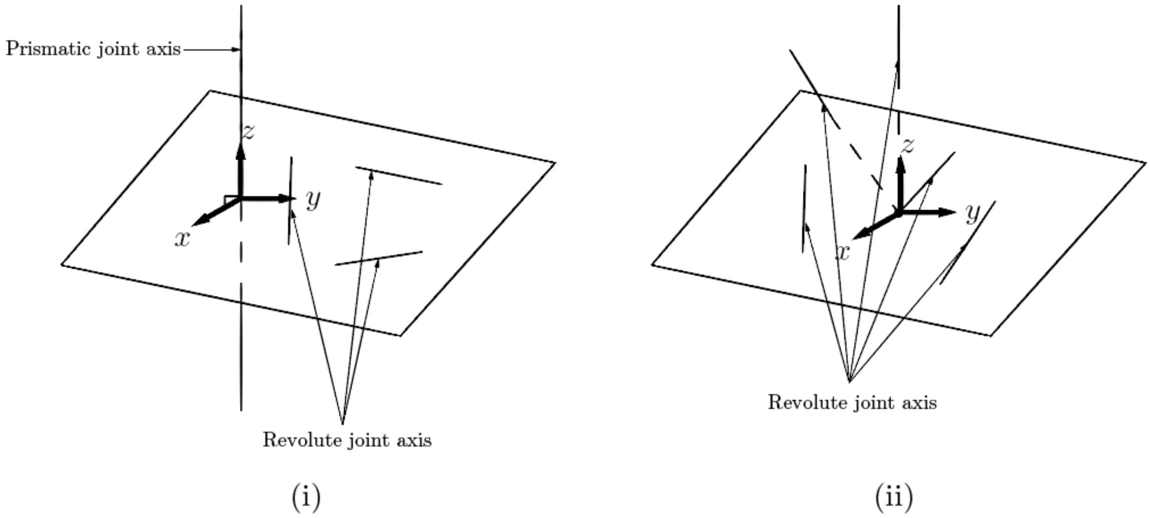


Figure 2: Figure for Problem 3.

$$\mathcal{V}_{s1}(\theta) : \omega_{s1} = \begin{bmatrix} w_{1x} \\ w_{1y} \\ 0 \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ v_{1z} \end{bmatrix}$$

$$\mathcal{V}_{s2}(\theta) : \omega_{s2} = \begin{bmatrix} w_{2x} \\ w_{2y} \\ 0 \end{bmatrix}, v_{s2} = \begin{bmatrix} 0 \\ 0 \\ v_{2z} \end{bmatrix}$$

$$\mathcal{V}_{s3}(\theta) : \omega_{s3} = \begin{bmatrix} w_{3x} \\ w_{3y} \\ 0 \end{bmatrix}, v_{s3} = \begin{bmatrix} 0 \\ 0 \\ v_{3z} \end{bmatrix}$$

$$\mathcal{V}_{s4}(\theta) : \omega_{s4} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, v_{s4} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore J_s(\theta) = \begin{bmatrix} w_{1x} & w_{2x} & w_{3x} & 0 \\ w_{1y} & w_{2y} & w_{3y} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ v_{1z} & v_{2z} & v_{3z} & 1 \end{bmatrix}$$

There is singularity because the rank of $J_s(\theta)$ is less than 4.

(b) Suppose that the the common point is coincident with the origin of the fixed frame and two other revolute joint axes are on the xy-plane of the fixed frame as shown in Figure 2-(ii).

$$\mathcal{V}_{s1}(\theta) : \omega_{s1} = \begin{bmatrix} w_{1x} \\ w_{1y} \\ w_{1z} \end{bmatrix}, v_{s1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\mathcal{V}_{s2}(\theta) : \omega_{s2} = \begin{bmatrix} w_{2x} \\ w_{2y} \\ w_{2z} \end{bmatrix}, v_{s2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\mathcal{V}_{s3}(\theta) : \omega_{s3} = \begin{bmatrix} w_{3x} \\ w_{3y} \\ w_{3z} \end{bmatrix}, v_{s3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
$$\mathcal{V}_{s4}(\theta) : \omega_{s4} = \begin{bmatrix} w_{4x} \\ w_{4y} \\ 0 \end{bmatrix}, v_{s4} = \begin{bmatrix} 0 \\ 0 \\ v_{4z} \end{bmatrix}$$
$$\mathcal{V}_{s5}(\theta) : \omega_{s5} = \begin{bmatrix} w_{5x} \\ w_{5y} \\ 0 \end{bmatrix}, v_{s5} = \begin{bmatrix} 0 \\ 0 \\ v_{5z} \end{bmatrix}$$

$$\therefore J_s(\theta) = \begin{bmatrix} w_{1x} & w_{2x} & w_{3x} & w_{4x} & w_{5x} \\ w_{1y} & w_{2y} & w_{3y} & w_{4y} & w_{5y} \\ w_{1z} & w_{2z} & w_{3z} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_{4z} & v_{4z} \end{bmatrix}$$

There is singularity because the rank of $J_s(\theta)$ is less than 5.

(c) There are only three types of singularity in the 6R PUMA-type arm.

(i) $\theta_5 = \pi/2$ or $3\pi/2$
- axes 4 and 6 are colinear.
(ii) $\theta_3 = 0$ or π
- axes 1,2,3,4,5, and 6 are intersecting a common line.
(iii) $m \cos\theta_2 + n\cos(\theta_2 + \theta_3) = 0$
- axes 4,5, and 6 are intersecting at a common point and this common point lies on the plane spanned by axes 1 and 2. (the case of Problem 3(b))

You can check the screw condition easily in each case.

In reference, we can check that there is no more type of singularity from Jacobian :

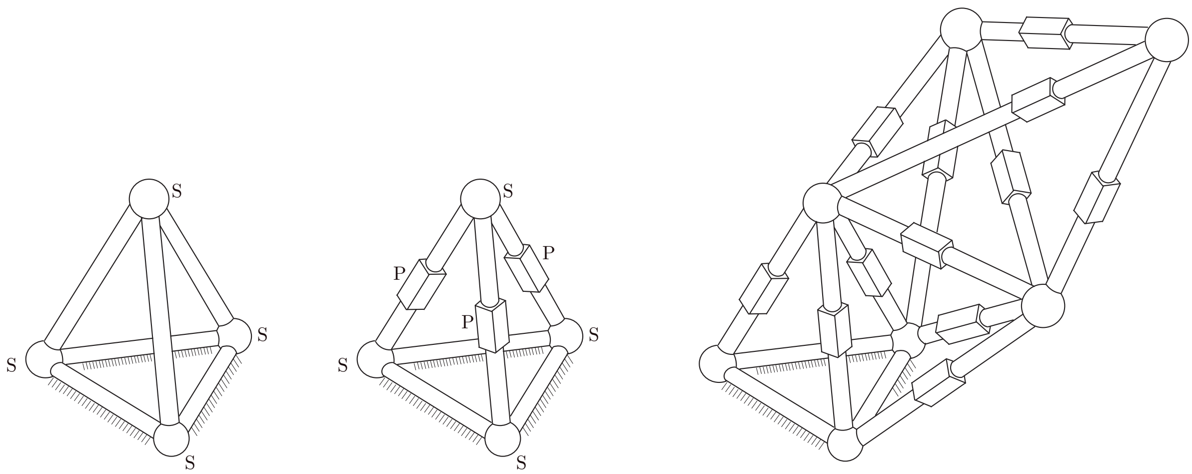
$$J(\theta) = \begin{bmatrix} 0 & 1 & 1 & 0 & \sin\theta_4 & \cos\theta_4\cos\theta_5 \\ \sin(\theta_2+\theta_3) & 0 & 0 & 1 & 0 & \sin\theta_5 \\ \cos(\theta_2+\theta_3) & 0 & 0 & 0 & \cos\theta_4 & -\sin\theta_4\cos\theta_5 \\ -m\cos\theta_2 & 0 & 0 & 0 & n\cos\theta_4 & -n\sin\theta_4 cos\theta_5 \\ -l\cos(\theta_2+\theta_3) & m\sin\theta_3 & 0 & 0 & 0 & 0 \\ l\sin(\theta_2+\theta_3) & m\cos\theta_3 & 0 & 0 & -n\sin\theta_4 & -n\cos\theta_4\cos\theta_5 \end{bmatrix}$$

$\det J(\theta) = mn(m\cos\theta_2 + n\cos(\theta_2+\theta_3))\sin\theta_3\cos\theta_5$

$\therefore \det J(\theta) = 0 \iff \theta_5 = \pi/2$ or $3\pi/2$, $\theta_3 = 0$ or π, $m\cos\theta_2 + n\cos(\theta_2+\theta_3) = 0$

M2794.002700 Introduction to Robotics
Final Examination
7-10 PM, June 13, 2017
CLOSED BOOK, CLOSED NOTES

Problem 1



(a) A single tetrahedron (b) A single tetrahedron with three struts actuated (c) Four tetrahedral modules are connected together with twelve struts actuated

Figure 1: Three types of structures assembled using tetrahedral modules

Figure 1 shows three types of structures made using tetrahedral modules. All of the legs (struts) are connected with spherical (S) joints at the nodes. Three base struts are always fixed to the ground (shown shaded).
(a) Use Grübler's formula to find the degrees of freedom of the single tetrahedron structure of Figure 1(a) (the struts are of fixed length). Does the result match your intuition? Explain.
(b) Suppose three struts of the single tetrahedron are linearly actuated by prismatic joints as in Figure 1(b). Determine the degrees of freedom of the mechanism using Grübler's formula, and explain if it agrees with your intuition.
(c) Suppose four tetrahedral modules are attached together to form the complex mechanism shown in Figure 1(c). Determine the degrees of freedom of this mechanism using Grübler's formula and explain if it agrees with your intuition. (*Hint*: Think of how the degrees of freedom changes whenver a tetrahedral is added).

Problem 2

A small robotic gripper is designed to pick up polygonal parts as shown in Figure 2. The parts are **regular polygons**: all interior angles have the same value β and the edges are of equal length ℓ. Assume there are four types of polygonal parts: triangles, squares, pentagons, and hexagons.

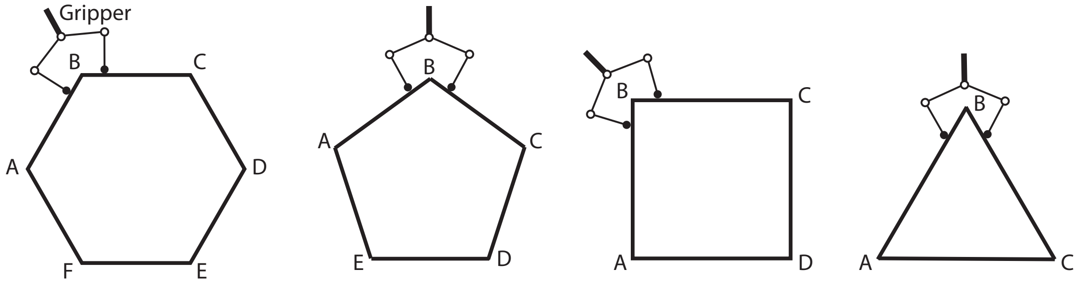


Figure 2: Picking up polygonal parts with a robotic gripper.

The gripper picks up a part using two contact points P_1, P_2 on adjacent edges (Figure 3): P_1 lies on edge AB, while P_2 lies on edge BC. Angles α and β are defined as shown in the figure. Assume frictional point contacts with friction coefficient μ; the friction cone angle θ as shown in the figure is then given by $\theta = \tan^{-1}(\mu)$.

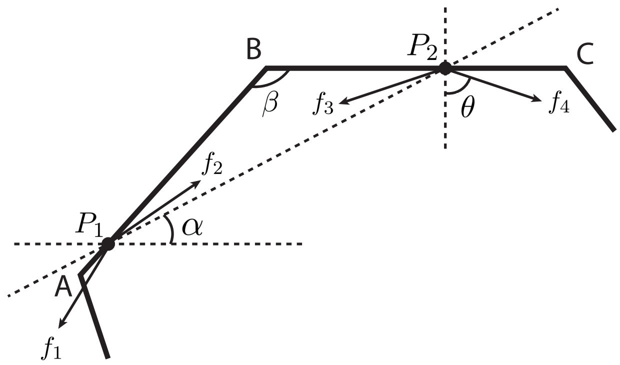


Figure 3: Gripper grasping a part at point contacts P_1 and P_2.

(a) Referring to Figure 3, derive a lower bound on θ for the grasp to be force closure. Express your lower bound in terms of the angles α and β.

(b) Observe α lies in the range $0 < \alpha < \pi - \beta$. What is the value of α such that the force closure can be achieved with the smallest friction coefficient μ? Express this optimal value for α in terms of the angle β.

(c) Assume that $\mu = \tan 58°$. Which of the four polygonal parts—the triangle, square, pentagon, and hexagon—can be picked up with the gripper?

Problem 3

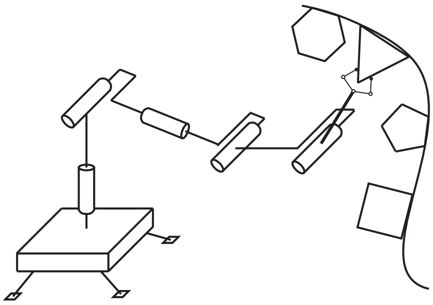


Figure 4: 5R robot for Problem 3.

The gripper of Problem 2 is now attached to the 5R robot of Figure 4. The zero position of the 5R robot is shown in Figure 5; {s} denotes the fixed frame, while {b} denotes the end-effector frame.

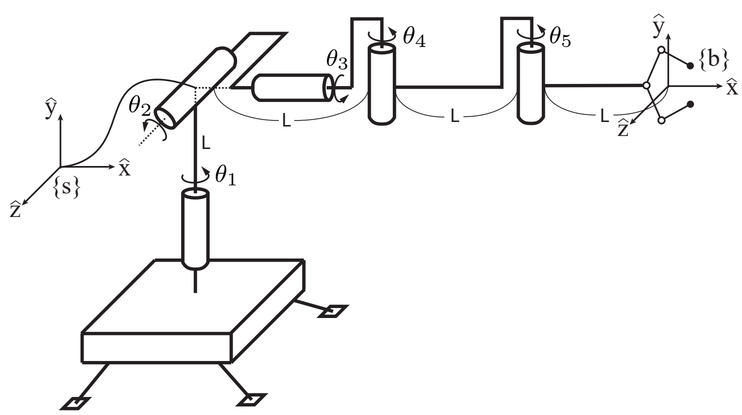


Figure 5: Zero position of the 5R robot.

(a) Express the forward kinematics as $T = e^{[S_1]\theta_1}e^{[S_2]\theta_2}e^{[S_3]\theta_3}e^{[S_4]\theta_4}e^{[S_5]\theta_5}M$, and derive M and S_i, $i = 1, \cdots, 5$.
(b) A polygonal part must be picked up in the configuration

$$T = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & (\sqrt{3}+\frac{1}{2})L \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2}L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Assume θ_3 is fixed to $\theta_3 = \frac{\pi}{2}$. How many inverse kinematics solutions does there exist? Derive all possible solutions θ_i in the range $-\pi < \theta_i \leq \pi$, $i = 1, 2, 4, 5$.
(c) Suppose a polygonal part must be pulled carefully from the wall in the configuration T given in (b). To do so, the robot must pull the part in the $-\hat{x}$ direction of the end-effector frame {b} while keeping the same orientation. Suppose the robot's maximum joint velocity is bounded by $\|\dot{\theta}\|^2 \leq 1$. What is the maximum possible linear velocity of the end-effector? (If you cannot solve this problem for the T given in part (b), then to receive partial credit, you may choose another more convenient T).

Problem 4
Recall the transformation between link frames using the Denavit-Hartenberg parameters:

- Revolute joints: $T_{i-1,i} = \text{Rot}(\hat{x}, \alpha_{i-1}) \cdot \text{Trans}(\hat{x}, a_{i-1}) \cdot \text{Trans}(\hat{z}, d_i) \cdot \text{Rot}(\hat{z}, \phi_i + \theta_i)$.

- Prismatic joints: $T_{i-1,i} = \text{Rot}(\hat{x}, \alpha_{i-1}) \cdot \text{Trans}(\hat{x}, a_{i-1}) \cdot \text{Trans}(\hat{z}, d_i + \theta_i) \cdot \text{Rot}(\hat{z}, \phi_i)$.

(a) Show that $T_{i-1,i}$ can be expressed as $T_{i-1,i}(\theta_i) = M_i e^{[\mathcal{S}]\theta_i}$ for both prismatic joints and revolute joints, and find $\mathcal{S} \in \mathbb{R}^6$ for each case.
(b) Given the D-H parameters and T_{nh} for an n-dof open chain robot, express the forward kinematics in the POE form $T_{0h} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \dots e^{[\mathcal{S}_n]\theta_n} M$.
(c) Assume $T_{0h} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} M_{0h}$, where $\mathcal{S}_1 = (0, -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0, 0)^T$, $\mathcal{S}_2 = (0, 0, 0, 0, 1, 0)^T$, and $M_{0h} \in SE(3)$ is

$$M_{0h} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 1 & 0 & 0 & \frac{\sqrt{3}}{2}L \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find, if they exist, the D-H parameters corresponding to the given $\mathcal{S}_1$ and $\mathcal{S}_2$, with $T_{2h} = I$, $d_1 = L$, and $\phi_1 = 0$. Otherwise, prove analytically that there are no feasible D-H parameters corresponding to the given POE parameters.
(d) Assume $T_{0h} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} M_{0h}$, where $\mathcal{S}_1 = (0, -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0, 0)^T$, $\mathcal{S}_2 = (0, 0, 0, 1, 0, 0)^T$, and M_{0h} is

$$M_{0h} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 1 & 0 & 0 & \frac{\sqrt{3}}{2}L \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find, if they exist, the D-H parameters corresponding to the given $\mathcal{S}_1$ and $\mathcal{S}_2$, with $T_{2h} = I$, $d_1 = L$, and $\phi_1 = 0$. Otherwise, prove analytically that there are no feasible D-H parameters correspoding to the given POE parameters.

Problem 5
The spatial six-dof open chain of Figure 6 is shown in its zero position, with space and end-effector frames chosen as shown. The pitch of the screw joint is $h = 1/2\pi$ (m/rad).

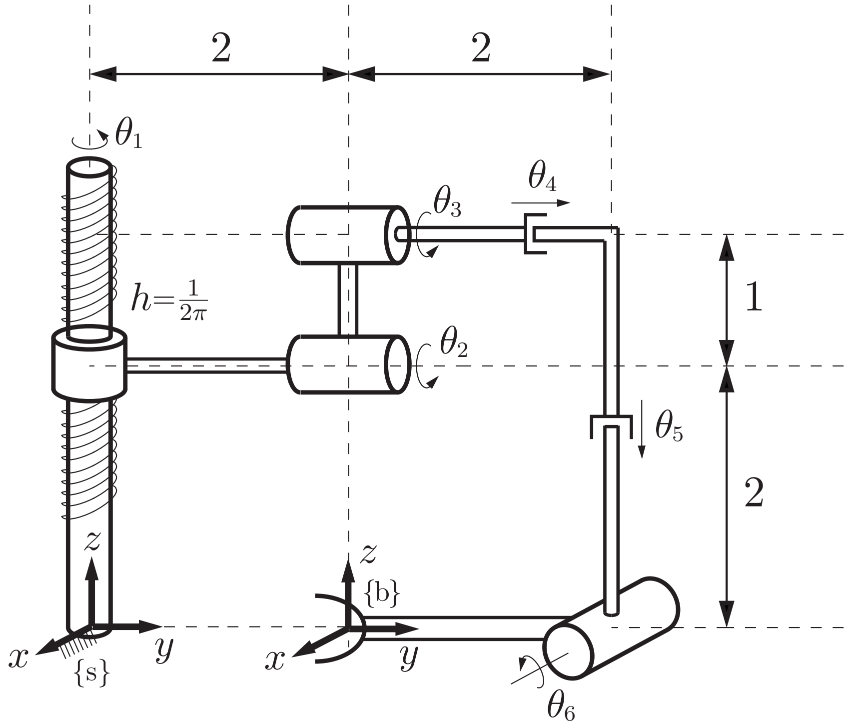


Figure 6: 6-DOF open chain for Problem 5

(a) When $\theta_1 = 2\pi$, $\theta_3 = -\pi/2$, $\theta_4 = 1$, $\theta_2 = \theta_5 = \theta_6 = 0$, derive the space Jacobian $J_s(\theta)$.
(b) Is the configuration in part (a) a singularity? If yes, explain which degrees of freedom of motion are lost by the end-effector.
(c) Assume the robot is in the same configuration as part (a). A force $f_b = (2, 1, 1)^T$, expressed in frame {b} coordinates, needs to be generated at the end-effector. What joint torques should be applied to generate this desired force?
(d) Now suppose θ_2 is fixed permanently to $\theta_2 = 0$. At the zero position, the robot's end-effector should generate some desired spatial velocity $V_d \in \mathbb{R}^6$ expressed in frame {b} coordinates. However, a solution $\dot{\theta} \in \mathbb{R}^5$ to $V_d = J_b(0)\dot{\theta}$ does not exist for the given V_d. Find the $\dot{\theta}$ that minimizes

$$f(\dot{\theta}) = \frac{1}{2}\left\|V_d - J_b(0)\dot{\theta}\right\|^2.$$

Express the optimal $\dot{\theta}$ in terms of $J_b(0)$ and V_d.

Problem 6

Figure 7 shows a 3R robot arm in its zero position. All lengths and angles are as shown in the figure. The three links all have the same shape and mass $m = 1$, and their **center-of-mass frames** $\{c_1\}$, $\{c_2\}$, and $\{c_3\}$ are attached at the respective link centers of mass. The 3×3 link inertia matrix with respect to the its center-of-mass frame is

$$\mathcal{I}_c = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The **link reference frames** $\{1\}$, $\{2\}$, and $\{3\}$ are attached at each joint as shown in the figure. Assume that there is no gravity.

(a) Let $\mathcal{G}_i$ be the 6×6 spatial inertia matrix for link i with respect to link frame $\{i\}$, $i = 1, 2, 3$. Find $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$.

(b) Let $\mathcal{V}_i$ be the body twist of link frame $\{i\}$ expressed in link frame $\{i\}$ coordinates, and $\dot{\mathcal{V}}_i$ its derivative, $i = 1, 2, 3$. The forward iteration equation for $\mathcal{V}_i$ is given by

$$\mathcal{V}_i = [\mathrm{Ad}_{T_{i,i-1}}]\mathcal{V}_{i-1} + \mathcal{A}_i\dot{\theta}_i,$$

where $\mathcal{A}_i$ is the screw axis for joint i, expressed in the link frame $\{i\}$. Derive this equation.

(c) Also recall that

$$\dot{\mathcal{V}}_i = [\mathrm{Ad}_{T_{i,i-1}}]\dot{\mathcal{V}}_{i-1} + [\mathrm{ad}_{\mathcal{V}_i}]\mathcal{A}_i\dot{\theta}_i + \mathcal{A}_i\ddot{\theta}_i,$$

and

$$\mathcal{F}_i = [\mathrm{Ad}_{T_{i+1,i}}]^T\mathcal{F}_{i+1} + \mathcal{G}_i\dot{\mathcal{V}}_i - [\mathrm{ad}_{\mathcal{V}_i}]^T\mathcal{G}_i\mathcal{V}_i,$$

and $\tau_i = \mathcal{F}_i^T\mathcal{A}_i$, where $\mathcal{F}_i$ is the wrench transmitted through joint i, expressed in frame $\{i\}$. Using the recursive inverse dynamics algorithm, find $m_{22}(0)$ and $m_{32}(0)$, the (2,2) and (3,2) entries of the mass matrix $M(0)$ when the robot is in its zero position.

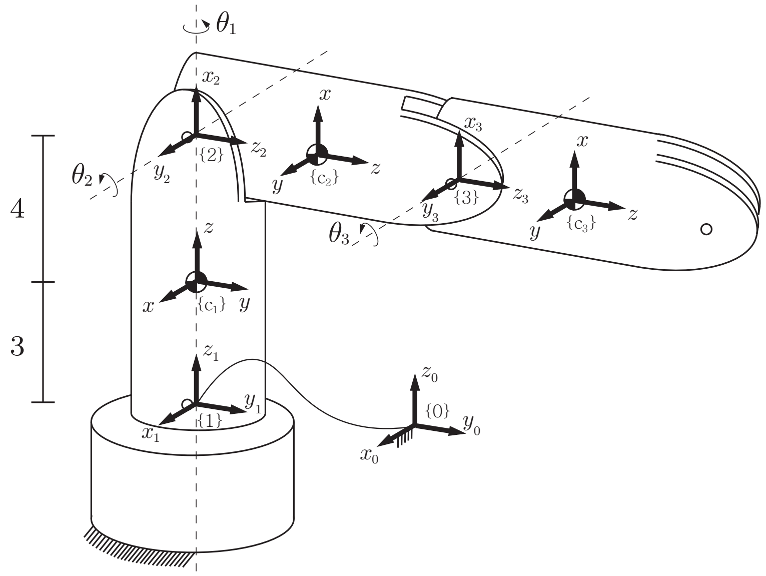
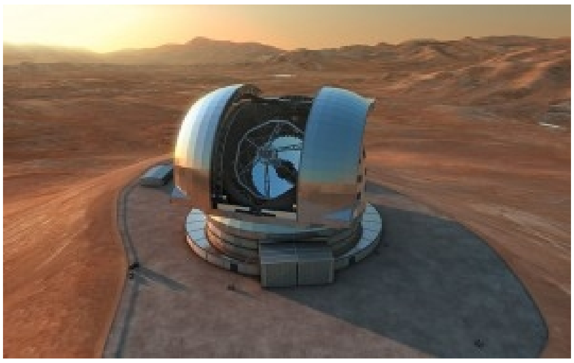


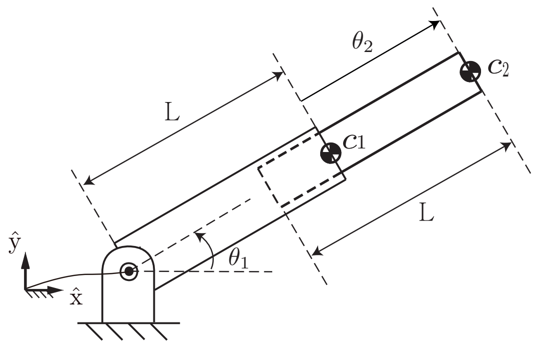
Figure 7: 3R robotic arm for Problem 5 shown in its zero position

Problem 7

Figure 8(a) shows a telescope operated by the European Southern Observatory in Chile. It can be modelled as an RP open chain as shown in Figure 8(b). The chain moves in the $\hat{x}$-$\hat{y}$ plane, with gravity g=10 acting in the -$\hat{y}$ direction. The two links are modelled as point masses $m_1 = m_2 = 1$ concentrated at the end of each link. The link length $L = 1$.



(a) Telescope for Problem 7.



(b) RP open chain model of telescope.

Figure 8: Figures for Problem 7

(a) Using the Lagrangian method, derive the dynamic equations for the telescope.
(b) For the constant input $\tau_1 = 0$, $\tau_2 = mg$, the telescope dynamics has the constant solution $\theta_1 = \pi/2$, $\theta_2 = L$. Linearize the dynamics about this solution and express it in the form $\dot{z} = Az + Bw$, where $z \in \mathbb{R}^4$, $w \in \mathbb{R}^2$, with $z_1 = \delta\theta_1$, $z_2 = \delta\theta_2$, $z_3 = \dot{z}_1$, $z_4 = \dot{z}_2$, $w_1 = \delta\tau_1$, $w_2 = \delta\tau_2$. If you are unable to obtain the dynamic equations for part (a), then explain the linearization process in as much detail as you can.
(c) Since earthquakes strike Chile frequently, a feedback controller must keep the telescope in stable equilibrium against small disturbances. For the linearized system obtained in part (b), design a PD control law of the form $\delta\tau_1 = k_p e + k_d \dot{e}$, where $e = \theta_d - \theta_1$, $\theta_d = \pi/2$, and $k_p = 35$, such that θ_1 is critically damped (that is, the telescope returns to the vertical position as fast as possible). If you are unable to obtain the linearized dynamics, then explain in as much detail as you can the critically damped PD control law for a standard mass-spring-damper system of the form $m\ddot{x} + b\dot{x} + kx = u$, where $x = \delta\theta_1$, $u = \delta\tau_1$.

M2794.002700 Introduction to Robotics
2017 Final Examination Solution

Problem 1 (50 Points)
(a) Since three base struts are fixed to the ground, the three base struts can be replaced with a ground. Using Grübler's formula for spatial mechanism yields:
N = 3 (links) + 1 (ground) = 4,
J = 5 (S joints),
$\sum f_i = 5 \times 3 = 15$,
dof $= 6(N-1-J) + \sum f_i = 6(4-1-5) + 15 = 3$.
However, since it is clear that a tetrahedron structure has a rigidity, the expected dof must be zero. The three additional degrees of freedom from Grübler's formula can be explained by the torsional rotations of the links about their respective axis.

(b) Considering the three prismatic joints, using Grübler's formula for spatial mechanism yields:
N = 6 (links) + 1 (ground) = 7,
J = 5 (S joints) + 3 (P joints),
$\sum f_i = 5 \times 3 + 3 \times 1 = 18$,
dof $= 6(N-1-J) + \sum f_i = 6(7-1-8) + 18 = 6$.
Intuitively, in three-dimensional space, the mechanism has three visible degrees of freedom at the end-effector. Grübler's formula overestimates the degrees of freedom also in this case due to the torsional rotations of the links.

(c) Suppose we have a result of Grübler's formula, dof $= 6(N-1-J) + \sum f_i$, before adding a tetrahedral module. When one more tetrahedral module is attached, we have five more S joints, three more P joints, and six more links. After adding one more module, using Grübler's formula for spatial mechanism yields:

$$\begin{aligned}
\text{dof} &= 6\{(N+6) - 1 - (J+8)\} + (\sum f_i + 5 \times 3 + 3 \times 1) \\
&= 6(N-1-J) + \sum f_i - 12 + 18 \\
&= 6(N-1-J) + \sum f_i + 6.
\end{aligned}$$

The degrees of freedom, according to Grübler's formula, increase by six each time a tetrahedral module is added. Since three more tetrahedral modules are added to a single tetrahedron serially (i.e., without making a closed loop of modules), we can expect Grübler's formula yields overall 24 degrees of freedom for the given mechanism. There must be, however, 12 internal degrees of freedom from torsional rotations of the links, the actual degrees of freedom is $24 - 12 = 12$.

Problem 2 (50 Points)
Figure 1 shows a gripper grasping a polygon with angle β. Angle α is defined as shown in the figure and the friction cone angle is $\theta = \tan^{-1}(\mu)$, where μ is the friction coefficient of the point contacts.

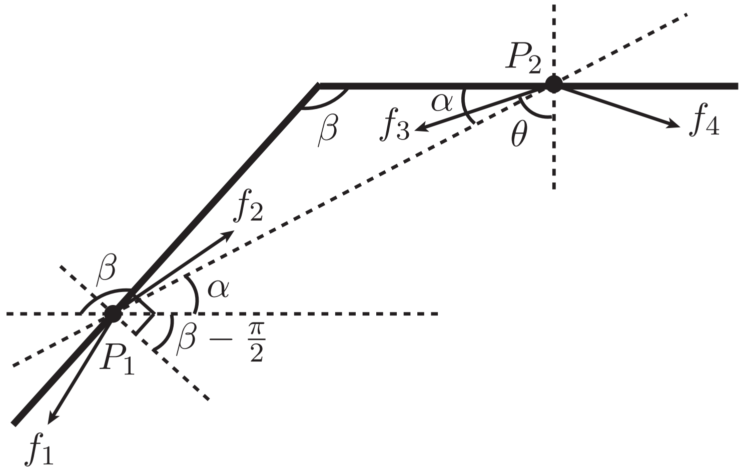


Figure 1: Gripper grasping a polygon with angle β

(a) According to Ngyuen's theorem, the line connecting the point contacts should lie inside both friction cones to achieve force closure. By geometry, the conditions for the line to lie inside both friction cones are:

- f_2 is above $\overline{P_1P_2} \implies \theta > \alpha + (\beta - \frac{\pi}{2})$,
- f_3 is above $\overline{P_1P_2} \implies \theta > \frac{\pi}{2} - \alpha$.

Therefore, the lower bound on θ for the grasp to be force closure is

$$\theta > \max\left(\alpha + (\beta - \frac{\pi}{2}), \frac{\pi}{2} - \alpha\right).$$

(b) Figure 2 shows the force closure conditions of θ plotted in $\theta - \alpha$ space, where the shaded area is the region of θ for force closure given α and β. The smallest friction coefficient corresponds to the smallest friction cone angle θ^*, which is achieved when

$$\theta^* = \alpha^* + \beta - \frac{\pi}{2} = \frac{\pi}{2} - \alpha^*,$$

where α^* is the optimal value for α. Therefore, the optimal value for α is

$$\alpha^* = \frac{\pi - \beta}{2}.$$

(c) The smallest friction cone angle θ^* that corresponds to the optimal α^* is

$$\theta^* = \frac{\beta}{2}.$$

Given β for each polygon, force closure will be achieved if $\theta > \theta^* = \frac{\beta}{2}$:

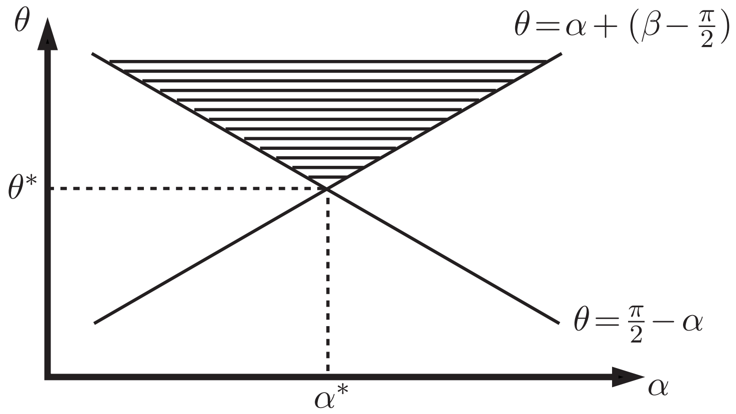


Figure 2: Force closure conditions of θ plotted

- Triangle: $\beta = 60°$, $\theta = 58° > \frac{\beta}{2} = 30° \implies$ Force closure,
- Square: $\beta = 90°$, $\theta = 58° > \frac{\beta}{2} = 45° \implies$ Force closure,
- Pentagon: $\beta = 108°$, $\theta = 58° > \frac{\beta}{2} = 54° \implies$ Force closure,
- Hexagon: $\beta = 120°$, $\theta = 58° < \frac{\beta}{2} = 60° \implies$ Not force closure.

Therefore, triangle, square, and pentagon can be picked up and hexagon cannot be picked up.

Problem 3 (60 Points)

(a) Revolute joint screw $S_i = [w_i, v_i]^T$ where w_i is axis of revolute joint, v_i is linear velocity of the origin of fixed frame, $v_i = -w_i \times q_i$, and q_i can be any point lying on its joint axis. Each of the joint screw S_i and the end-effector configuration at zero-position M can be derived as follows:

i	w_i	q_i	v_i
1	$(0, 1, 0)$	$(0, 0, 0)$	$(0, 0, 0)$
2	$(0, 0, 1)$	$(0, 0, 0)$	$(0, 0, 0)$
3	$(1, 0, 0)$	$(0, 0, 0)$	$(0, 0, 0)$
4	$(0, 1, 0)$	$(L, 0, 0)$	$(0, 0, L)$
5	$(0, 1, 0)$	$(2L, 0, 0)$	$(0, 0, 2L)$

$$M = \begin{bmatrix} 1 & 0 & 0 & 3L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

since the orientation of the end-effector is the same as fixed frame and the position is lying on the x-axis of fixed frame.

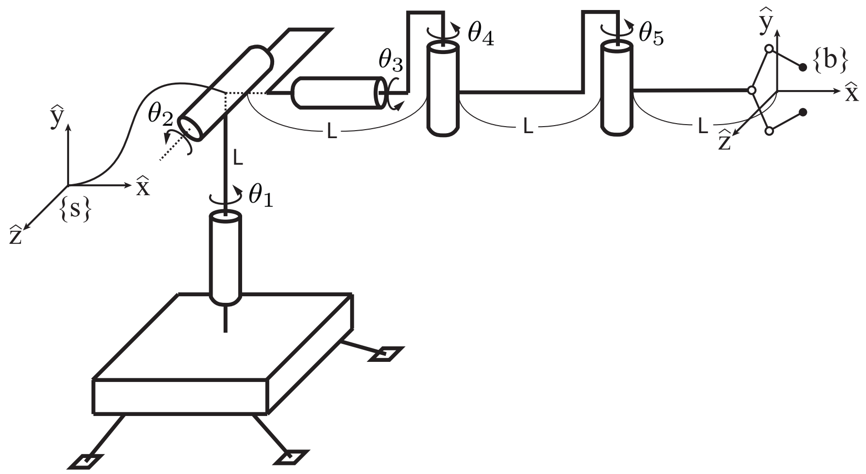


Figure 3: Zero position of the mining 5R manipulator.

(b) Since $\theta_3 = \frac{\pi}{2}$, $\theta_2, \theta_4, \theta_5$ cannot change the z-component of the position of {b} frame, p_z, but only θ_1 determines. Thus, given $p_z = 0$ means that $\theta_1 = 0, \pi$. Given fixed $\theta_3 = \frac{\pi}{2}$, there are 4 solutions generally. There are two pairs for each combination of (θ_1, θ_2) and (θ_4, θ_5), which leads to the total number of combinations, $(\theta_1, \theta_2, \theta_4, \theta_5)$, is $2 \times 2 = 4$.

Basically, (θ_4, θ_5) combinations are elbow-up and elbow-down as Fig 4. Imagine $\theta_1 = \pi$ and $\theta_2 = \pi$, then you can easily figure out that the manipulator pose is perfectly the same as the home position except the direction of joint axes $\theta_2, \theta_4, \theta_5$ (Later, it will be shown that $\theta_1 \neq \pi$ because of the orientation of the end-effector.). It is easy to start with position of joint 5, J_5. The position can be derived by translating end-effector towards $-\hat{x}$ axis of {b} frame:

$$\hat{x}_b = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \\ 0 \end{bmatrix}, \quad p_5 = p - L\hat{x}_b = \begin{bmatrix} \sqrt{3}L \\ 0 \\ 0 \end{bmatrix},$$

where p_5 is the position of J_5 in Fig 4. Then we can compute θ_4 with cosine law and all other θ_i's can be determined by geometrical calculation.

	θ_1	θ_2	θ_4	θ_5
1	0	$\frac{1}{6}\pi$	$-\frac{1}{3}\pi$	$\frac{1}{2}\pi$
2	0	$-\frac{1}{6}\pi$	$\frac{1}{3}\pi$	$\frac{1}{6}\pi$
3	π	$-\frac{5}{6}\pi$	$-\frac{1}{3}\pi$	$-\frac{1}{6}\pi$
4	π	$\frac{5}{6}\pi$	$\frac{1}{3}\pi$	$-\frac{1}{2}\pi$

However, y-axis and z-axis of the body frame, $\hat{y}_b$ and $\hat{z}_b$, are not consistent with given end-effector configuration, T, when $\theta_1 = \pi$. Thus, the answer should eliminate the cases of $\theta_1 = \pi$ as follows:

	θ_1	θ_2	θ_4	θ_5
1	0	$\frac{1}{6}\pi$	$-\frac{1}{3}\pi$	$\frac{1}{2}\pi$
2	0	$-\frac{1}{6}\pi$	$\frac{1}{3}\pi$	$\frac{1}{6}\pi$

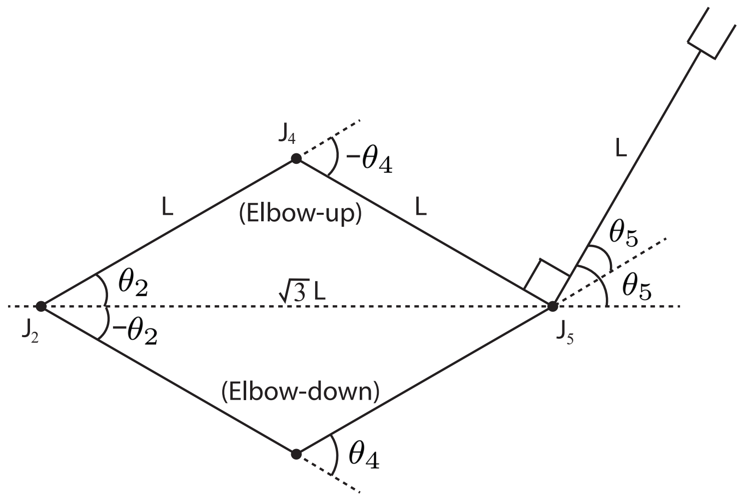


Figure 4: Elbow-up and elbow-down for inverse kinematics.

(c) Using the solution of (b) such that $0 \le \theta_2 \le \frac{\pi}{2}$, we can derive the body jacobian as follows:

i	w_i
1	$(\frac{\sqrt{3}}{2}, 0, -\frac{1}{2})$
2	$(0, 1, 0)$
3	$(\frac{1}{2}, 0, \frac{\sqrt{3}}{2})$
4	$(0, 1, 0)$
5	$(0, 1, 0)$

To minimize scratching, desired velocity of the end-effector is $w_b = 0$, $v_b = [-v, 0, 0]^T$, pulling in $-\hat{x}_b$ direction without rotation. Since $w_b = \sum_i w_i \dot{\theta}_i = 0$,

$$\dot{\theta}_1 = \dot{\theta}_3 = 0,$$
$$\dot{\theta}_2 + \dot{\theta}_4 + \dot{\theta}_5 = 0.$$

To deal with $v_b = [-v, 0, 0]^T$, it is recommended to solve the problem in b frame. The velocity components v_i of the body jacobian of joint 2, 4, 5 can be derived as follows:

i	w_i	q_i	v_i
2	$(0,1,0)$	$(-(1+\frac{\sqrt{3}}{2})L, 0, -\frac{3}{2}L)$	$(\frac{3}{2}L, 0, -(1+\frac{\sqrt{3}}{2})L)$
4	$(0,1,0)$	$(-L, 0, -L)$	$(L, 0, -L)$
5	$(0,1,0)$	$(-L, 0, 0)$	$(0, 0, -L)$

Note that $(v_b)_z = 0 = -L\dot{\theta}_4 - L\dot{\theta}_5 - (1+\frac{\sqrt{3}}{2})L\dot{\theta}_2$. Substituting $\dot{\theta}_2 + \dot{\theta}_4 + \dot{\theta}_5 = 0$ from $\sum_i w_i \dot{\theta}_i = 0$ leads to $\dot{\theta}_2 = 0$ and $\dot{\theta}_4 = -\dot{\theta}_5$. The problem is to maximize $(v_b)_x = -L\dot{\theta}_4 - \frac{3}{2}L\dot{\theta}_2$ under the constraint $\|\dot{\theta}\|^2 \leq 1$. The answer should be,

$$\max_{\|\dot{\theta}\|^2 \leq 1} \|(v_b)_x\| = \max_{\dot{\theta}_4^2 \leq \frac{1}{2}} \| - L\dot{\theta}_4\| = \frac{L}{\sqrt{2}}.$$

Problem 4 (70 Points)
(a) For a revolute joint,

$$\begin{aligned} T_{i-1,i} &= \mathrm{Rot}(\hat{x}, \alpha_{i-1}) \cdot \mathrm{Trans}(\hat{x}, a_{i-1}) \cdot \mathrm{Trans}(\hat{z}, d_i) \cdot \mathrm{Rot}(\hat{z}, \phi_i + \theta_i) \\ &= \mathrm{Rot}(\hat{x}, \alpha_{i-1}) \cdot \mathrm{Trans}(\hat{x}, a_{i-1}) \cdot \mathrm{Trans}(\hat{z}, d_i) \cdot \mathrm{Rot}(\hat{z}, \phi_i) \cdot \mathrm{Rot}(\hat{z}, \theta_i) \\ &= M_i e^{[\mathcal{S}]\theta_i}. \end{aligned}$$

Therefore,

$$\begin{aligned} M_i &= \mathrm{Rot}(\hat{x}, \alpha_{i-1}) \cdot \mathrm{Trans}(\hat{x}, a_{i-1}) \cdot \mathrm{Trans}(\hat{z}, d_i) \cdot \mathrm{Rot}(\hat{z}, \phi_i) \\ \mathcal{S} &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T. \end{aligned}$$

For a prismatic joint,

$$\begin{aligned} T_{i-1,i} &= \mathrm{Rot}(\hat{x}, \alpha_{i-1}) \cdot \mathrm{Trans}(\hat{x}, a_{i-1}) \cdot \mathrm{Trans}(\hat{z}, d_i + \theta_i) \cdot \mathrm{Rot}(\hat{z}, \phi_i) \\ &= \mathrm{Rot}(\hat{x}, \alpha_{i-1}) \cdot \mathrm{Trans}(\hat{x}, a_{i-1}) \cdot \mathrm{Rot}(\hat{z}, \phi_i) \cdot \mathrm{Trans}(\hat{z}, d_i + \theta_i) \\ &= \mathrm{Rot}(\hat{x}, \alpha_{i-1}) \cdot \mathrm{Trans}(\hat{x}, a_{i-1}) \cdot \mathrm{Rot}(\hat{z}, \phi_i) \cdot \mathrm{Trans}(\hat{z}, d_i) \cdot \mathrm{Trans}(\hat{z}, \theta_i) \\ &= M_i e^{[\mathcal{S}]\theta_i}. \end{aligned}$$

Therefore,

$$\begin{aligned} M_i &= \mathrm{Rot}(\hat{x}, \alpha_{i-1}) \cdot \mathrm{Trans}(\hat{x}, a_{i-1}) \cdot \mathrm{Rot}(\hat{z}, \phi_i) \cdot \mathrm{Trans}(\hat{z}, d_i) \\ \mathcal{S} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T. \end{aligned}$$

(b) From the result of (a),

$$\begin{aligned} T_{0h} &= T_{01} T_{12} \cdots T_{n-1,n} T_{nh} \\ &= M_1 e^{[\mathcal{A}_1]\theta_1} M_2 e^{[\mathcal{A}_2]\theta_2} \cdots M_n e^{[\mathcal{A}_n]\theta_n} T_{nh} \\ &= \left(M_1 e^{[\mathcal{A}_1]\theta_1} M_1^{-1}\right) \left(M_1 M_2 e^{[\mathcal{A}_2]\theta_2} M_2^{-1} M_1^{-1}\right) \\ &\quad \cdots \left(M_1 \cdots M_n e^{[\mathcal{A}_n]\theta_n} M_n^{-1} \cdots M_1^{-1}\right) M_1 \cdots M_n T_{nh} \\ &= e^{M_1 [\mathcal{A}_1] M_1^{-1} \theta_1} e^{(M_1 M_2)[\mathcal{A}_2](M_1 M_2)^{-1}\theta_2} \cdots e^{(M_1 \cdots M_n)[\mathcal{A}_n](M_1 \cdots M_n)^{-1}\theta_n} M_1 \cdots M_n T_{nh} \\ &= e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{S}_1 &= \mathrm{Ad}_{M_1}(\mathcal{A}_1) \\ \mathcal{S}_2 &= \mathrm{Ad}_{M_1 M_2}(\mathcal{A}_2) \\ &\vdots \\ \mathcal{S}_n &= \mathrm{Ad}_{M_1 \cdots M_n}(\mathcal{A}_n) \\ M &= M_1 \cdots M_n T_{nh}. \end{aligned}$$

(c) Suppose that there exists $M_1 \in SE(3)$ satisfying $S_1 = Ad_{M_1}(S)$,
where $M_1 = R_x(\alpha_0)T_x(a_0)T_z(d_1)R_z(\phi_1)$, $S = (0,0,1,0,0,0)^T$.
(Since $S_1 = (0, -\frac{1}{2}, \frac{\sqrt{3}}{2}, 0, 0, 0)^T$, we know that the joint 1 is a revolute joint.)

$$M_1 = \begin{bmatrix} C_{\phi 1} & -S_{\phi 1} & 0 & a_0 \\ S_{\phi 1}C_{\alpha 0} & C_{\phi 1}C_{\alpha 0} & -S_{\alpha 0} & -d_1 S_{\alpha 0} \\ S_{\phi 1}S_{\alpha 0} & C_{\phi 1}S_{\alpha 0} & C_{\alpha 0} & d_1 C_{\alpha 0} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}.$$

$$S_1 = \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix} S, \ where \ S = (0,0,1,0,0,0)^T.$$

$$\rightarrow (0, -\tfrac{1}{2}, \tfrac{\sqrt{3}}{2})^T \text{= the } 3^{rd} \text{ column of R, } (0,0,0)^T \text{=the } 3^{rd} \text{ column of } [p]R.$$

$$\begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -S_{\alpha 0} \\ C_{\alpha 0} \end{bmatrix} \ \rightarrow \ \alpha_0 = \pi/6.$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = [p]R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ a_0 C_{\alpha 0} \\ a_0 S_{\alpha 0} \end{bmatrix} \ \rightarrow \ a_0 = 0.$$

$$\rightarrow \alpha_0 = \pi/6, \ a_0 = 0, \ d_1 = L, \ \phi_1 = 0.$$

Suppose that there exists $M_2 \in SE(3)$ satisfying $S_2 = Ad_{M_1 M_2}(S)$,
where $M_2 = R_x(\alpha_1)T_x(a_1)T_z(d_2)R_z(\phi_2)$, $S = (0,0,0,0,0,1)^T$.
(Since $S_2 = (0,0,0,0,1,0)^T$, we know that the joint 2 is a prismatic joint.)

$$T_{0h} = e^{[S_1]\theta_1} e^{[S_2]\theta_2} M_{0h} = M_1 e^{[S]\theta_1} M_1{}^{-1} M_1 M_2 e^{[S]\theta_2} (M_1 M_2)^{-1} M_{0h}$$

$$\rightarrow (M_1 M_2)^{-1} M_{0h} = T_{2h} = I.$$

$$\rightarrow M_1 M_2 = M_{0h}.$$

Let's check $S_2 = Ad_{M_1 M_2}(S)$, where $S = (0,0,0,0,0,1)^T$.

$$S_2 = Ad_{M_1 M_2}(S) = Ad_{M_{0h}}(S) = \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix} S, \ where \ M_{0h} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}.$$

$$\rightarrow S_2 = the \ 6^{th} \ column \ of \ \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix} = (0,0,0,0,1,0)^T \rightarrow Checked!$$

$$
\begin{aligned}
M_1 M_2 &= M_{0h} \\
M_2 &= {M_1}^{-1} M_{0h} \\
&= T_z(-d_1) R_x(-\alpha_0) M_{0h} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & -\frac{L}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2}L \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L \\ 1 & 0 & 0 & \frac{\sqrt{3}}{2}L \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{4}L \\ \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} & -\frac{3}{4}L \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} C_{\phi 2} & -S_{\phi 2} & 0 & a_1 \\ S_{\phi 2}C_{\alpha 1} & C_{\phi 2}C_{\alpha 1} & -S_{\alpha 1} & -d_2 S_{\alpha 1} \\ S_{\phi 2}S_{\alpha 1} & C_{\phi 2}S_{\alpha 1} & C_{\alpha 1} & d_2 C_{\alpha 1} \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

$$\rightarrow \alpha_1 = 4\pi/3,\ a_1 = 0,\ d_2 = 3L/2,\ \phi_2 = -\pi/2.$$

* You can solve it by visualizing the given system.

(d) You can easily show the existence of M_1 from (c).
Let's check the existence of M_2.
S_2 should be equivalent to $Ad_{M_1 M_2}(S)$, where $S = (0,0,0,0,0,1)^T$.
$Ad_{M_1 M_2}(S)$= the 6^{th} column of $\begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix} = (0,0,0,0,1,0)^T \neq S_2$
$\rightarrow$ The D-H parameters correspoding to the given system does not exist.

Problem 5 (50 Points)
Figure 5 shows the open chain when $\theta_1 = 2\pi$, $\theta_3 = -\pi/2$, $\theta_4 = 1$, $\theta_2 = \theta_5 = \theta_6 = 0$.

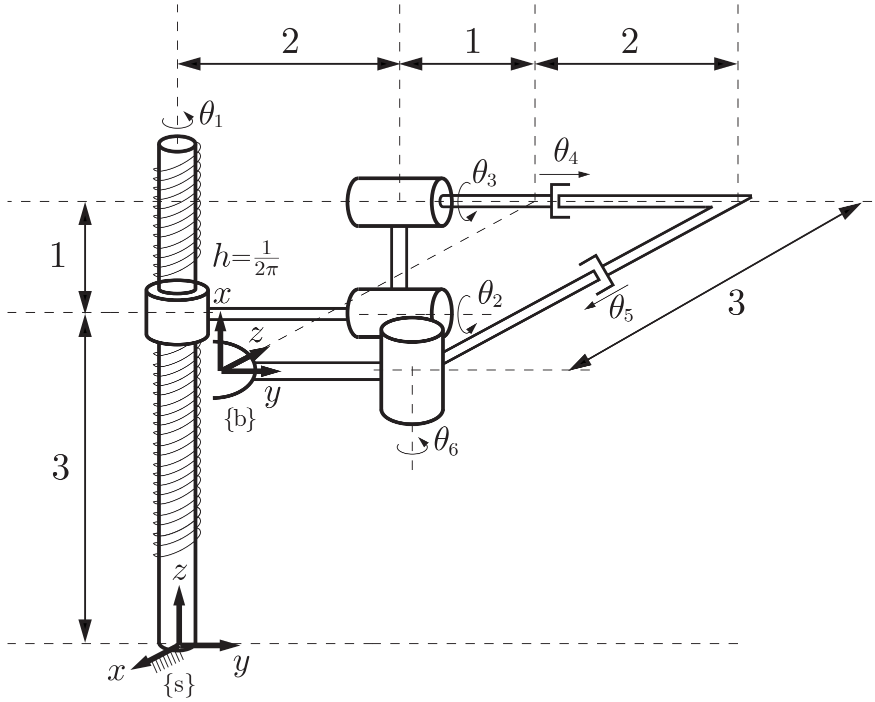


Figure 5: Open chain when $\theta_1 = 2\pi$, $\theta_3 = -\pi/2$, $\theta_4 = 1$, $\theta_2 = \theta_5 = \theta_6 = 0$

(a) For joints 2 to 6, set w_i as its joint axis and set $v_i = -w_i \times q_i$, where q_i is an arbitrary point on its joint axis. For joint 1, set w_1 as its joint axis and set $v_1 = -w_1 \times q_1 + hw_1$, where h is the pitch of the joint.

i	w_i	q_i	v_i
1	$(0,0,1)$	$(0,0,0)$	$(0,0,1/2\pi)$
2	$(0,1,0)$	$(0,0,3)$	$(-3,0,0)$
3	$(0,1,0)$	$(0,0,4)$	$(-4,0,0)$
4	$(0,0,0)$	$\times$	$(0,1,0)$
5	$(0,0,0)$	$\times$	$(1,0,0)$
6	$(0,0,1)$	$(3,5,0)$	$(5,-3,0)$

Then, the space Jacobian $J_s(\theta)$ is

$$J_s(\theta) = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -3 & -4 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 & 0 & -3 \\ 1/2\pi & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Since the first row of $J_s(\theta)$ is all zero, rank of $J_s(\theta)$ is smaller than 6. Therefore, the configuration in part (a) is a singularity. For arbitrary $\dot{\theta}$, it can be seen that

$$V_s = \begin{bmatrix} w_s \\ v_s \end{bmatrix} = J_s(\theta)\dot{\theta} = \begin{bmatrix} 0 \\ w_y \\ w_z \\ v_x \\ v_y \\ v_z \end{bmatrix},$$

where w_y, w_z, v_x, v_y, and v_z are arbitrary values. Therefore, the end-effector cannot rotate about x-axis with respect to frame {s}. Note that the rank of $J_s(\theta)$ is 5.
(c) From Figure 5, the rotation matrix from frame {s} to frame {b} is

$$R_{sb} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Therefore, desired force f_b can be expressed in frame {s} coordinates as

$$f_s = R_{sb} f_b = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Then, the end-effector will generate moment of

$$m_s = r_s \times f_s = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \\ 6 \end{bmatrix},$$

where r_s is the position of the end-effector expressed in frame {s} coordinates. Then the spatial force expressed in frame {s} coordinates is $\mathcal{F}_s = (2, -10, 6, -1, 1, 2)^T$. Therefore, the joint torques that should be applied are

$$\tau = J_s(\theta)^T \mathcal{F}_s = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1/2\pi \\ 0 & 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 5 & -3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -10 \\ 6 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 + \frac{1}{\pi} \\ -7 \\ -6 \\ 1 \\ -1 \\ -2 \end{bmatrix}.$$

(d) Since θ_2 is fixed permanently to $\theta_2 = 0$, joint 2 can be seen as a part of link between joint 1 and joint 3. At the zero position, the body Jacobian $J_b(0)$ is

$$J_b(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -2 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1/2\pi & 0 & 0 & -1 & -2 \end{bmatrix} \in \mathbb{R}^{6 \times 5}.$$

The rank of $J_b(0)$ is 5, which is full rank. $f(\dot{\theta})$ can be expanded to

$$\begin{aligned}
f(\dot{\theta}) &= \frac{1}{2}\left\|V_d - J_b(0)\dot{\theta}\right\|^2 \\
&= \frac{1}{2}\left(V_d - J_b(0)\dot{\theta}\right)^T\left(V_d - J_b(0)\dot{\theta}\right) \\
&= \frac{1}{2}\left(V_d^T V_d - 2V_d^T J_b(0)\dot{\theta} + \dot{\theta}^T J_b(0)^T J_b(0)\dot{\theta}\right).
\end{aligned}$$

Now we apply first order necessary condition to $f(\dot{\theta})$ to find the optimal $\dot{\theta}^*$:

$$\begin{aligned}
\left.\frac{\partial f(\dot{\theta})}{\partial \dot{\theta}}\right|_{\dot{\theta}=\dot{\theta}^*} &= \frac{1}{2}\left(2J_b(0)^T J_b(0)\dot{\theta}^* - 2J_b(0)^T V_d\right) \\
&= J_b(0)^T J_b(0)\dot{\theta}^* - J_b(0)^T V_d \\
&= 0.
\end{aligned}$$

Since the rank of $J_b(0)$ is 5, rank of $J_b(0)^T J_b(0) \in \mathbb{R}^{5\times 5}$ is 5 and thus invertible. Therefore, the optimal $\dot{\theta}^*$ is

$$\dot{\theta}^* = \left(J_b(0)^T J_b(0)\right)^{-1} J_b(0)^T V_d.$$

Problem 6 (60 Points)

(a) We can find $\mathcal{G}_1$ from $\mathcal{G}_1 = [Ad_{T_{c_1 1}}]^T \mathcal{G}_{c_1} [Ad_{T_{c_1 1}}]$, where $\mathcal{G}_{c_1} = \begin{bmatrix} \mathcal{I}_c & 0 \\ 0 & mI \end{bmatrix}$.

We can also derive $[Ad_{T_{c_1 1}}] = \begin{bmatrix} I & 0 \\ [p_{c_1 1}] & I \end{bmatrix}$ by $R_{c_1 1} = I, p_{c_1 1} = (0, 0, -3)$. Therefore,

$$
\begin{aligned}
\mathcal{G}_1 &= [Ad_{T_{c_1 1}}]^T \mathcal{G}_{c_1} [Ad_{T_{c_1 1}}] \\
&= \begin{bmatrix} \mathcal{I}_c - m[p_{c_1 1}]^2 & -m[p_{c_1 1}] \\ m[p_{c_1 1}] & mI \end{bmatrix} \\
&= \begin{bmatrix} 13 & 0 & 0 & 0 & -3 & 0 \\ 0 & 13 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

and $\mathcal{G}_1 = \mathcal{G}_2 = \mathcal{G}_3$.

(b)

$$
\begin{aligned}
[\mathcal{V}_i] &= T_{0i}^{-1} \dot{T}_{0i} \\
&= (T_{0,i-1} T_{i-1,i})^{-1} (T_{0,i-1} \dot{T}_{i-1,i}) \\
&= T_{i-1,i}^{-1} T_{0,i-1}^{-1} (\dot{T}_{0,i-1} T_{i-1,i} + T_{0,i-1} \dot{T}_{i-1,i}) \\
&= T_{i-1,i}^{-1} T_{0,i-1}^{-1} \dot{T}_{0,i-1} T_{i-1,i} + T_{i-1,i}^{-1} T_{0,i-1}^{-1} T_{0,i-1} \dot{T}_{i-1,i} \\
&= T_{i-1,i}^{-1} T_{0,i-1}^{-1} \dot{T}_{0,i-1} T_{i-1,i} + T_{i-1,i}^{-1} \dot{T}_{i-1,i} \\
&= [Ad_{T_{i,i-1}}(\mathcal{V}_{i-1})] + [\mathcal{A}_i \dot{\theta}_i]
\end{aligned}
$$

$\therefore \mathcal{V}_i = [Ad_{T_{i,i-1}}] \mathcal{V}_i + \mathcal{A}_i \dot{\theta}_i$

(c)

Link 1: $\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1(t)$. Frame $\{\mathbf{1}\}$ cannot move linearly, $v_1 = \dot{v}_1 = 0$.

$$\mathcal{V}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \dot{\mathcal{V}}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Link 2:

$$R_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -s_2 & 0 & c_2 \\ c_2 & 0 & s_2 \end{bmatrix}.$$

Then, we can find T_{12} and T_{21}.

$$T_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -s_2 & 0 & c_2 & 0 \\ c_2 & 0 & s_2 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow T_{21} = \begin{bmatrix} 0 & -s_2 & c_2 & -7c_2 \\ 1 & 0 & 0 & 0 \\ 0 & c_2 & s_2 & -7s_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathrm{Ad}_{T_{21}}] = \begin{bmatrix} R_{21} & 0 \\ [p_{21}]R_{21} & R_{21} \end{bmatrix}$$

Therefore,

$$\mathcal{V}_2 = [Ad_{T_{21}}]\mathcal{V}_1 + \mathcal{A}_2\dot{\theta}_2 = \begin{bmatrix} c_2\dot{\theta}_1 \\ 0 \\ s_2\dot{\theta}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\theta}_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_2\dot{\theta}_1 \\ \dot{\theta}_2 \\ s_2\dot{\theta}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\dot{\mathcal{V}}_2 = \begin{bmatrix} c_2\ddot{\theta}_1 - s_2\dot{\theta}_1\dot{\theta}_2 \\ \ddot{\theta}_2 \\ s_2\ddot{\theta}_1 + c_2\dot{\theta}_1\dot{\theta}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where, $s_2 = \sin\theta_2, c_2 = \cos\theta_2$.

Link 3:

$$R_{23} = \begin{bmatrix} c_3 & 0 & s_3 \\ 0 & 1 & 0 \\ -s_3 & 0 & c_3 \end{bmatrix}$$

Then, we can find T_{23} and T_{32}.

$$T_{23} = \begin{bmatrix} c_3 & 0 & s_3 & 0 \\ 0 & 1 & 0 & 0 \\ -s_3 & 0 & c_3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow T_{32} = \begin{bmatrix} c_3 & 0 & -s_3 & 7s_3 \\ 0 & 1 & 0 & 0 \\ s_3 & 0 & c_3 & -7c_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathrm{Ad}_{T_{32}}] = \begin{bmatrix} R_{32} & 0 \\ [p_{32}]R_{32} & R_{32} \end{bmatrix}$$

Therefore,

$$\mathcal{V}_3 = [Ad_{T_{32}}]\mathcal{V}_2 + \mathcal{A}_3\dot{\theta}_3 = \begin{bmatrix} c_{23}\dot{\theta}_1 \\ \dot{\theta}_2 \\ s_{23}\dot{\theta}_1 \\ 7c_3\dot{\theta}_2 \\ -7c_2\dot{\theta}_1 \\ 7s_3\dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dot{\theta}_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \\ s_{23}\dot{\theta}_1 \\ 7c_3\dot{\theta}_2 \\ -7c_2\dot{\theta}_1 \\ 7s_3\dot{\theta}_2 \end{bmatrix}.$$

$$\dot{\mathcal{V}}_3 = \begin{bmatrix} c_{23}\ddot{\theta}_1 - s_{23}\dot{\theta}_1\dot{\theta}_2 - s_{23}\dot{\theta}_1\dot{\theta}_3 \\ \ddot{\theta}_2 + \ddot{\theta}_3 \\ s_{23}\ddot{\theta}_1 + c_{23}\dot{\theta}_1\dot{\theta}_2 + c_{23}\dot{\theta}_1\dot{\theta}_3 \\ 7c_3\ddot{\theta}_2 - 7s_3\dot{\theta}_2\dot{\theta}_3 \\ -7c_2\ddot{\theta}_1 + 7s_2\dot{\theta}_1\dot{\theta}_2 \\ 7s_3\ddot{\theta}_2 + 7c_3\dot{\theta}_2\dot{\theta}_3 \end{bmatrix}$$

where, $s_3 = \sin\theta_3, c_3 = \cos\theta_3$, $s_{23} = \sin(\theta_2 + \theta_3)$, $c_{23} = \cos(\theta_2 + \theta_3)$.
From the result, we can derive dynamic equation of the robot by backward iteration.
We assume that the robot is in zero position. (i.e. $\theta = 0$)
We only need to calculate components of mass matrix, so we can set $\dot{\theta} = 0$

$$\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{V}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dot{\mathcal{V}}_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dot{\mathcal{V}}_2 = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \dot{\mathcal{V}}_3 = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 + \ddot{\theta}_3 \\ 0 \\ 7\ddot{\theta}_2 \\ -7\ddot{\theta}_1 \\ 0 \end{bmatrix}.$$

Link 3: There is no other external force-moment on the link 3 except applied by link 2.

$$\mathcal{F}_3 = \mathcal{G}_3\dot{\mathcal{V}}_3 - [\mathrm{ad}_{\mathcal{V}_3}]^T\mathcal{G}_3\mathcal{V}_3$$

$$\mathcal{F}_3 = \begin{bmatrix} 13 & 0 & 0 & 0 & -3 & 0 \\ 0 & 13 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 + \ddot{\theta}_3 \\ 0 \\ 7\ddot{\theta}_2 \\ -7\ddot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 34\ddot{\theta}_1 \\ 34\ddot{\theta}_2 + 13\ddot{\theta}_3 \\ 0 \\ 10\ddot{\theta}_2 + 3\ddot{\theta}_3 \\ -10\ddot{\theta}_1 \\ 0 \end{bmatrix}$$

Link 2:

$$\mathcal{F}_2 = \mathcal{G}_2\dot{\mathcal{V}}_2 - [\mathrm{ad}_{\mathcal{V}_2}]^T\mathcal{G}_2\mathcal{V}_2 + [\mathrm{Ad}_{T_{32}}]^T\mathcal{F}_3$$

$$\begin{aligned} \mathcal{F}_2 &= \begin{bmatrix} 13 & 0 & 0 & 0 & -3 & 0 \\ 0 & 13 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + [\mathrm{Ad}_{T_{32}}]^T \begin{bmatrix} 34\ddot{\theta}_1 \\ 34\ddot{\theta}_2 + 13\ddot{\theta}_3 \\ 0 \\ 10\ddot{\theta}_2 + 3\ddot{\theta}_3 \\ -10\ddot{\theta}_1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} ? \\ 117\ddot{\theta}_2 + 34\ddot{\theta}_3 \\ ? \\ ? \\ ? \\ ? \end{bmatrix} \end{aligned}$$

$\tau_2 = \mathcal{F}_2^T\mathcal{A}_2$ where $\mathcal{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T$, so we only need to calculate the second component of $\mathcal{F}_2$

$$\tau_2 = 117\ddot{\theta}_2 + 34\ddot{\theta}_3$$

Therefore, $m_{22}(0) = 117, m_{32}(0) = 34$

Problem 7 (60 Points)
(a) The position and velocity of link 1 are given by

$$p_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} Lcos\theta_1 \\ Lsin\theta_1 \end{bmatrix}, v_1 = \begin{bmatrix} -Lsin\theta_1 \\ Lcos\theta_1 \end{bmatrix} \dot{\theta}_1,$$

while those of link 2 are given by

$$p_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} (L+\theta_2)cos\theta_1 \\ (L+\theta_2)sin\theta_1 \end{bmatrix}, v_2 = \begin{bmatrix} -(L+\theta_2)sin\theta_1 & cos\theta_1 \\ (L+\theta_2)cos\theta_1 & sin\theta_1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}.$$

The link kinetic engergy terms $\mathcal{K}_1$ and $\mathcal{K}_2$ are

$$\begin{aligned} \mathcal{K}_1 &= \frac{1}{2}m(\dot{x_1}^2 + \dot{y_1}^2) = \frac{1}{2}mL^2\dot{\theta_1}^2 \\ \mathcal{K}_2 &= \frac{1}{2}m(\dot{x_2}^2 + \dot{y_2}^2) = \frac{1}{2}m((L+\theta_2)^2\dot{\theta_1}^2 + \dot{\theta_2}^2) \\ \mathcal{K} &= \mathcal{K}_1 + \mathcal{K}_2 = \frac{1}{2}I\dot{\theta_1}^2 + \frac{1}{2}m\dot{\theta_2}^2, where\ I = mL^2 + m(L+\theta_2)^2, \end{aligned}$$

and the system potential energy terms $\mathcal{P}$ are

$$\mathcal{P} = mgLs_1 + mg(L+\theta_2)s_1 = mgs_1(2L+\theta_2).$$

The Euler-Lagrange equations for this system are of the form

$$\tau_i = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i} - \frac{\partial \mathcal{L}}{\partial \theta_i},\ i = 1, 2.$$

Then, we have

$$\begin{aligned} \tau_1 &= I\ddot{\theta}_1 + 2m(L+\theta_2)\dot{\theta}_1\dot{\theta}_2 + mg(2L+\theta_2)c_1 \\ \tau_2 &= m\ddot{\theta}_2 - m(L+\theta_2)\dot{\theta_1}^2 + mgs_1,\ where\ I = mL^2 + m(L+\theta_2)^2. \end{aligned}$$

(b) Let $x_1 = \theta_1$, $x_2 = \theta_2$, $x_3 = \dot{x_1}$, $x_3 = \dot{x_2}$.
The constant solution is given by

$$\bar{x_1} = \pi/2,\ \bar{x_2} = L,\ \bar{x_3} = 0,\ \bar{x_4} = 0,\ \bar{\tau_1} = 0,\ \bar{\tau_2} = mg.$$

Let's linearize the dynamics about this solution.

$$\begin{aligned} \delta\dot{x}_1 &= \delta x_3 \\ \delta\dot{x}_2 &= \delta x_4 \\ \delta\dot{x}_3 &= \frac{3g}{5L}\delta x_1 + \frac{1}{5mL^2}\delta\tau_1 \\ \delta\dot{x}_4 &= \frac{1}{m}\delta\tau_2. \end{aligned}$$

The matrix form of the linearized dynamics is expressed by

$$\dot{z} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{3g}{5L} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{5mL^2} & 0 \\ 0 & \frac{1}{m} \end{bmatrix} w.$$

(c) We need to use the dynamics related to θ_1. Since $\delta x_3 = \dot{\delta x_1}$, the equation

$$\dot{\delta x_3} = \frac{3g}{5L}\delta x_1 + \frac{1}{5mL^2}\delta\tau_1$$

is the independent dynamics about δx_1 which is operated by $\delta\tau_1$.

$$\begin{aligned} e &= \theta_d - \theta_1 = \theta_d - (\theta_d + \delta\theta_1) = -\delta\theta_1 \\ \delta\tau_1 &= k_p e + k_d \dot{e} = -k_p\delta\theta_1 - k_d\delta\dot{\theta}_1 \\ \delta\ddot{\theta}_1 - \frac{3g}{5L}\delta\theta_1 &= \frac{1}{5mL^2}\delta\tau_1 = \frac{1}{5mL^2}(-k_p\delta\theta_1 - k_d\delta\dot{\theta}_1) \end{aligned}$$

$$\delta\ddot{\theta}_1 + \frac{k_d}{5mL^2}\delta\dot{\theta}_1 + (\frac{k_p}{5mL^2} - \frac{3g}{5L})\delta\theta_1 = 0.$$

For critical damping, $(\frac{k_d}{5mL^2})^2 - 4(\frac{k_p}{5mL^2} - \frac{3g}{5L})$ should be zero.
Thus, $k_d = 10$.