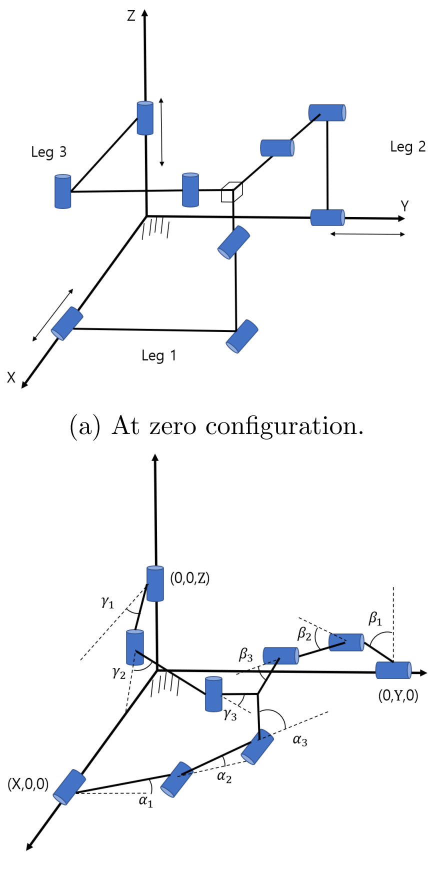
M2794.0027 Introduction to Robotics
Midterm Examination
11-12pm, 2020 May 14
CLOSED BOOK, CLOSED NOTES

Problem 1 (25 points)
(a) (10 points) The $3 \times PRRR$ mechanism of Figure 1(a) is shown in its zero configuration. The mechanism is symmetrically designed: all links are of length 1, and the joint axes are either parallel or orthogonal as indicated in the figure. Use Grübler's formula to calculate its degrees of freedom.
(b) (15 points) Looking at Figure 1(b), intuitively it should be clear that this mechanism can indeed move. Derive a set of constraint equations that describe its configuration space. Using a counting argument, determine the degrees of freedom of this mechanism. (*Hint*: The mechanism can be thought of as three $PRRR$ open chains whose tips intersect orthogonally at a common point. Label the joint variables for the three chains as $(X, \alpha_1, \beta_1, \gamma_1)$, $(Y, \alpha_2, \beta_2, \gamma_2)$, $(Z, \alpha_3, \beta_3, \gamma_3)$, and derive an appropriate set of constraint equations. You may further assume that $0 < \alpha_1 + \alpha_2 + \alpha_3 < \pi$, $0 < \beta_1 + \beta_2 + \beta_3 < \pi$, $0 < \gamma_1 + \gamma_2 + \gamma_3 < \pi$.



(a) At zero configuration.

(b) Joint variables.

Figure 1: The $3 \times PRRR$ mechanism of Problem 1.

Problem 2 (20 points)
A 4×4 box is grasped at the corners by four two-fingered grippers as shown in Figure 2. Assume

the eight point contacts are frictionless, and each gripper can only apply normal forces of equal magnitude (that is, the normal forces at A and B always have the same magnitude, and similarly for the normal forces at C-D, E-F, and G-H). Is this grasp force closure? Carefully explain your answer.

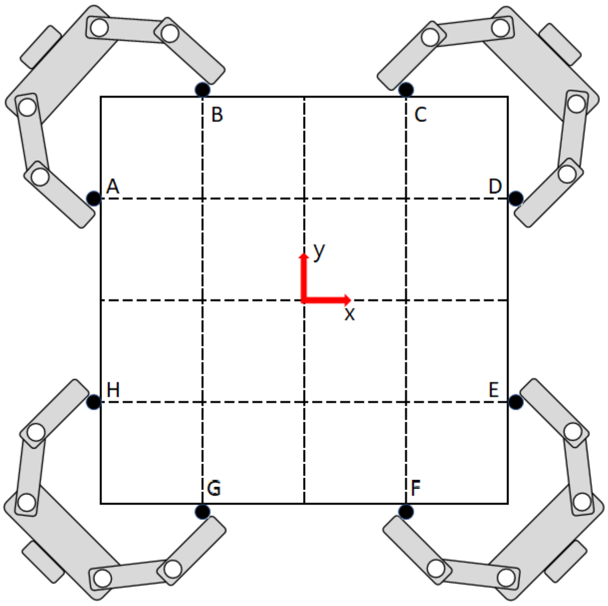


Figure 2: Grasped object for Problem 2.

Problem 3 (25 points)
The 3R ball-throwing robot of Figure 3 is shown in its zero configuration. All links are of length 1, and the arrows on the joint axes indicate the direction of positive rotation. Reference frame $\{ball\}$ is attached to the ball as shown, and matches the end-effector frame $\{3\}$. Suppose the ball is released when $(\theta_1, \theta_2, \theta_3) = (\frac{\pi}{2}, \frac{\pi}{2}, \pi)$, with joint velocities $(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3) = (1, 1, 1)$ (in units of rad/sec). Assuming $t = 0$ when the ball is released, find $T_{0,ball}(t)$ as a function of time t. You may ignore gravity, and assume the linear and angular acceleration of the ball is zero at $t = 0$.

Problem 4 (30 points)

Figure 4(a) shows the Staubli 6R robot, with its zero configuration shown in Figure 4(b).

(a) Draw appropriate link reference frames, and derive the Denavit-Hartenberg parameters for the first three joints only.

(b) The forward kinematics can be expressed in the form

$$T_{06}(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} M_4 e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M_6.$$

Find M_4, M_6, and the $\mathcal{S}_i$, $i = 1, \ldots, 6$.

(c) Letting $T_{46} = e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M_6$, find $\dot{T}_{46} T_{46}^{-1}$.

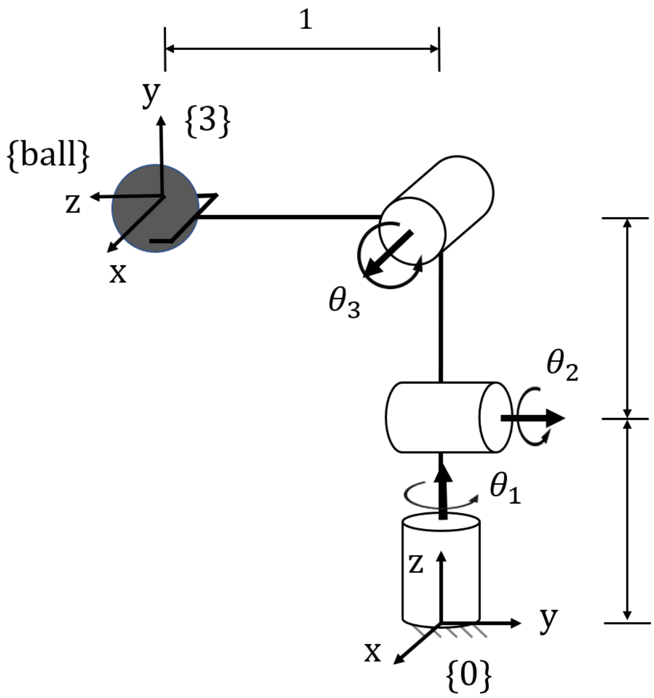
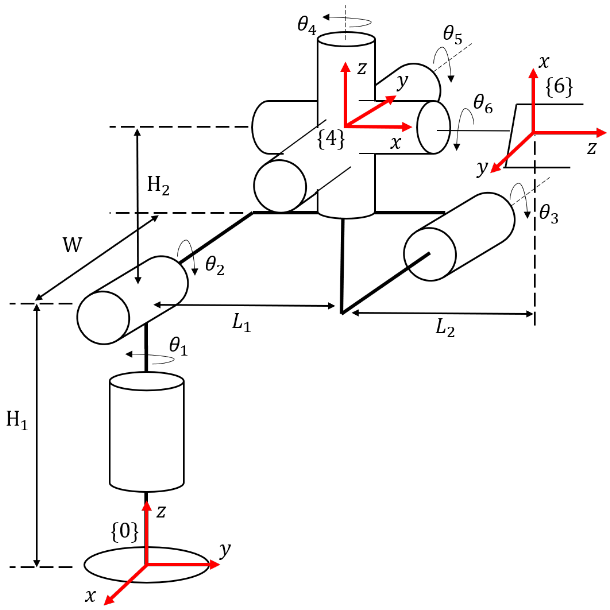


Figure 3: Figure for Problem 3.

(a) 6R open chain robot, tx40.



(b) Frame assignments in zero configuration.

Figure 4: Figure for Problem 4

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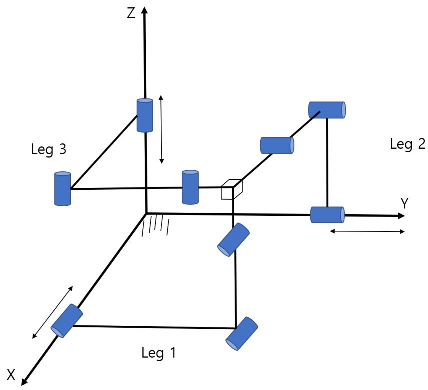
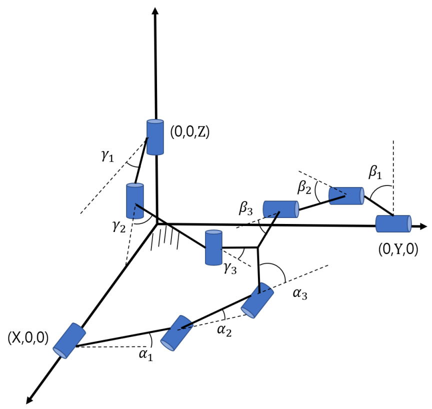
Problem 1 (25 points)

(a) (10 points) The $3 \times PRRR$ mechanism of Figure 1(a) is shown in its zero configuration. The mechanism is symmetrically designed: all links are of length 1, and the joint axes are either parallel or orthogonal as indicated in the figure. Use Grübler's formula to calculate its degrees of freedom.
(b) (15 points) Looking at Figure 1(b), intuitively it should be clear that this mechanism can indeed move. Derive a set of constraint equations that describe its configuration space. Using a counting argument, determine the degrees of freedom of this mechanism. (*Hint*: The mechanism can be thought of as three $PRRR$ open chains whose tips intersect orthogonally at a common point. Label the joint variables for the three chains as $(X, \alpha_1, \beta_1, \gamma_1)$, $(Y, \alpha_2, \beta_2, \gamma_2)$, $(Z, \alpha_3, \beta_3, \gamma_3)$, and derive an appropriate set of constraint equations. You may further assume that $0 < \alpha_1 + \alpha_2 + \alpha_3 < \pi$, $0 < \beta_1 + \beta_2 + \beta_3 < \pi$, $0 < \gamma_1 + \gamma_2 + \gamma_3 < \pi$.



(a) At zero configuration.



(b) Joint variables.

Figure 1: The $3 \times PRRR$ mechanism of Problem 1.

Solution

(a) Regarding a PR joint as a two-dof cylindrical (C) joint, we have a total of $J = 9$ joints (three C joints, six R joints), $N = 8$ links (including ground), and $\sum f_i = 3 \times 2 + 6 \times 1 = 12$ joint degrees of freedom. From Grübler's formula,

$$\text{dof} = 6(N - 1 - J) + \sum f_i = 0.$$

(b) The Cartesian (x, y, z) tip coordinates for the three legs are

$$\begin{aligned}
\text{Leg 1} &= (X, \cos\alpha_1 + \cos\alpha_{12} + \cos\alpha_{123}, \sin\alpha_1 + \sin\alpha_{12} + \sin\alpha_{123}) \\
\text{Leg 2} &= (\sin\beta_1 + \sin\beta_{12} + \sin\beta_{123}, Y, \cos\beta_1 + \cos\beta_{12} + \cos\beta_{123}) \\
\text{Leg 3} &= (\cos\gamma_1 + \cos\gamma_{12} + \cos\gamma_{123}, \sin\gamma_1 + \sin\gamma_{12} + \sin\gamma_{123}, Z),
\end{aligned}$$

where we use the notation $\alpha_{12} = \alpha_1 + \alpha_2$, $\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3$, etc. Since the three leg tips meet at the same point, we have the following six constraint equations:

$$\begin{aligned}
X &= \sin\beta_1 + \sin\beta_{12} + \sin\beta_{123} &&= \cos\gamma_1 + \cos\gamma_{12} + \cos\gamma_{123} \\
Y &= \cos\alpha_1 + \cos\alpha_{12} + \cos\alpha_{123} &&= \sin\gamma_1 + \sin\gamma_{12} + \sin\gamma_{123} \\
Z &= \sin\alpha_1 + \sin\alpha_{12} + \sin\alpha_{123} &&= \cos\beta_1 + \cos\beta_{12} + \cos\beta_{123}.
\end{aligned}$$

The tip links also meet orthogonally; calculating the direction of each tip link,

$$\begin{aligned}
\text{Leg 1 tip direction} &= (0, \cos\alpha_{123}, \sin\alpha_{123}) \\
\text{Leg 2 tip direction} &= (\sin\beta_{123}, 0, \cos\beta_{123}) \\
\text{Leg 3 tip direction} &= (\cos\gamma_{123}, \sin\gamma_{123}, 0),
\end{aligned}$$

and the orthogonality condition implies

$$\begin{aligned}
\sin\alpha_{123}\cos\beta_{123} &= 0 \\
\sin\beta_{123}\cos\gamma_{123} &= 0 \\
\cos\alpha_{123}\sin\gamma_{123} &= 0.
\end{aligned}$$

From the problem statement we have $0^\circ < \alpha_{123}, \beta_{123}, \gamma_{123} < 180^\circ$, implying that

$$\begin{aligned}
\alpha_1 + \alpha_2 + \alpha_3 &= 90^\circ \\
\beta_1 + \beta_2 + \beta_3 &= 90^\circ \\
\gamma_1 + \gamma_2 + \gamma_3 &= 90^\circ.
\end{aligned}$$

Since there are a total of twelve joint variables and nine equality constraints, the mechanism must therefore have $12 - 9 = 3$ degrees of freedom.

Problem 2 (20 points)
A 4×4 box is grasped at the corners by four two-fingered grippers as shown in Figure 2. Assume the eight point contacts are frictionless, and each gripper can only apply normal forces of equal magnitude (that is, the normal forces at A and B always have the same magnitude, and similarly for the normal forces at C-D, E-F, and G-H). Is this grasp force closure? Carefully explain your answer.

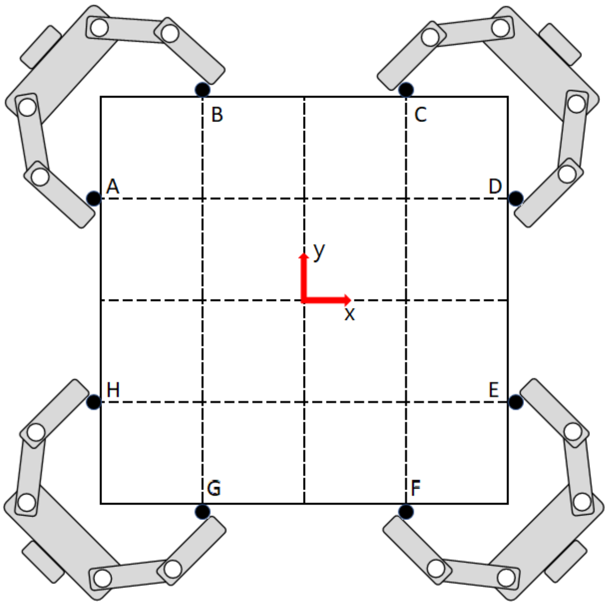


Figure 2: Grasped object for Problem 2.

Solution

Let $\hat{\mathrm{n}}_A$ be the inward-pointing unit vector normal to the grasped object at contact A. The contact force at A is then $\mathrm{f}_A = f_A \hat{\mathrm{n}}_A$, where $f_A \geq 0$ is the magnitude of f_A. Let r_A be the vector from the reference frame origin to A. From the force-moment equilibrium equations,

$$\begin{bmatrix} \hat{\mathrm{n}}_A & \hat{\mathrm{n}}_B & \cdots & \hat{\mathrm{n}}_G & \hat{\mathrm{n}}_H \\ \mathrm{r}_A \times \hat{\mathrm{n}}_A & \mathrm{r}_B \times \hat{\mathrm{n}}_B & \cdots & \mathrm{r}_G \times \hat{\mathrm{n}}_G & \mathrm{r}_H \times \hat{\mathrm{n}}_H \end{bmatrix} \begin{bmatrix} f_A \\ f_B \\ \vdots \\ f_G \\ f_H \end{bmatrix} = \begin{bmatrix} -\mathrm{f}_{\mathrm{ext}} \\ \\ -\mathrm{m}_{\mathrm{ext}} \end{bmatrix},$$

where $\mathrm{f}_{\mathrm{ext}}$ and $\mathrm{m}_{\mathrm{ext}}$ respectively denote the two-dimensional external force and one-dimensional (scalar) external moment applied to the object. From the problem statement we have $f_A = f_B$, $f_C = f_D$, $f_E = f_F$, and $f_G = f_H$, so that the force-moment equilibrium equation can be rewritten

$$\begin{bmatrix} \hat{\mathrm{n}}_A & \hat{\mathrm{n}}_B & \cdots & \hat{\mathrm{n}}_G & \hat{\mathrm{n}}_H \\ \mathrm{r}_A \times \hat{\mathrm{n}}_A & \mathrm{r}_B \times \hat{\mathrm{n}}_B & \cdots & \mathrm{r}_G \times \hat{\mathrm{n}}_G & \mathrm{r}_H \times \hat{\mathrm{n}}_H \end{bmatrix} \begin{bmatrix} f_A \\ f_A \\ \vdots \\ f_G \\ f_G \end{bmatrix} = \begin{bmatrix} -\mathrm{f}_{\mathrm{ext}} \\ \\ -\mathrm{m}_{\mathrm{ext}} \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} \hat{\mathrm{n}}_A + \hat{\mathrm{n}}_B & \hat{\mathrm{n}}_C + \hat{\mathrm{n}}_D & \hat{\mathrm{n}}_E + \hat{\mathrm{n}}_F & \hat{\mathrm{n}}_G + \hat{\mathrm{n}}_G \\ \mathrm{r}_A \times \hat{\mathrm{n}}_A + \mathrm{r}_B \times \hat{\mathrm{n}}_B & \mathrm{r}_C \times \hat{\mathrm{n}}_C + \mathrm{r}_D \times \hat{\mathrm{n}}_D & \mathrm{r}_E \times \hat{\mathrm{n}}_E + \mathrm{r}_F \times \hat{\mathrm{n}}_F & \mathrm{r}_G \times \hat{\mathrm{n}}_G + \mathrm{r}_H \times \hat{\mathrm{n}}_H \end{bmatrix} \begin{bmatrix} f_A \\ f_C \\ f_E \\ f_G \end{bmatrix} = \begin{bmatrix} -\mathrm{f}_{\mathrm{ext}} \\ \\ -\mathrm{m}_{\mathrm{ext}} \end{bmatrix}.$$

Choosing the reference frame at the object center as shown in the figure, the force-moment equilibrium condition becomes

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_A \\ f_C \\ f_E \\ f_G \end{bmatrix} = \begin{bmatrix} -\mathrm{f}_{\mathrm{ext}} \\ \\ -\mathrm{m}_{\mathrm{ext}} \end{bmatrix}.$$

Recalling that the grasp is force closure if and only if, for any arbitrary $\mathrm{f}_{\mathrm{ext}} \in \mathbb{R}^2$ and $\mathrm{m}_{\mathrm{ext}} \in \mathbb{R}$, there exists nonnegative values for (f_A, f_C, f_E, f_G) such that the above equality is always satisfied, this grasp is clearly not force closure; the last row of zeros implies that arbitrary external moments cannot be resisted by the grasp.

Problem 3 (25 points)
The 3R ball-throwing robot of Figure 3 is shown in its zero configuration. All links are of length 1, and the arrows on the joint axes indicate the direction of positive rotation. Reference frame {ball} is attached to the ball as shown, and matches the end-effector frame {3}. Suppose the ball is released when $(\theta_1, \theta_2, \theta_3) = (\frac{\pi}{2}, \frac{\pi}{2}, \pi)$, with joint velocities $(\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3) = (1, 1, 1)$ (in units of rad/sec). Assuming $t = 0$ when the ball is released, find $T_{0,ball}(t)$ as a function of time t. You may ignore gravity, and assume the linear and angular acceleration of the ball is zero at $t = 0$.

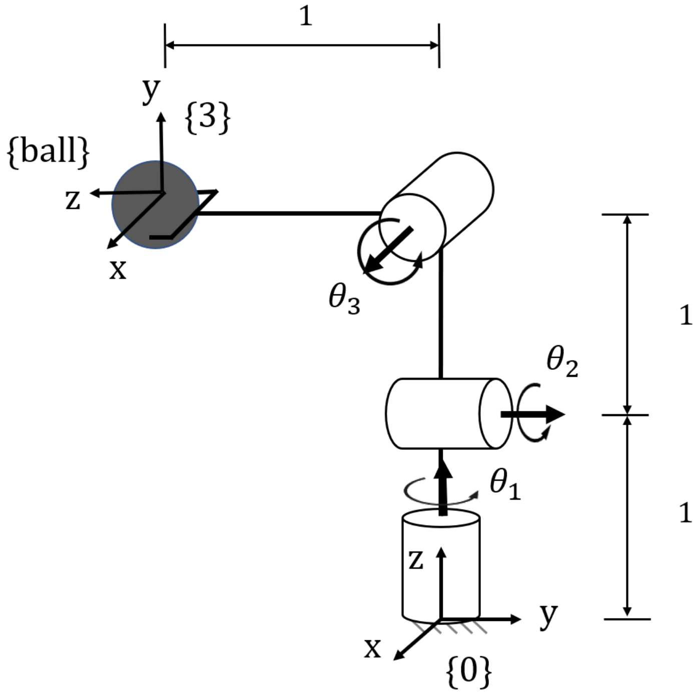


Figure 3: $3R$ robot for Problem 3.

Solution
The robot is at the configuration $(\theta_1(0), \theta_2(0), \theta_3(0)) = (\frac{\pi}{2}, \frac{\pi}{2}, \pi)$ when the ball is released at $t = 0$ (see Figure 4).

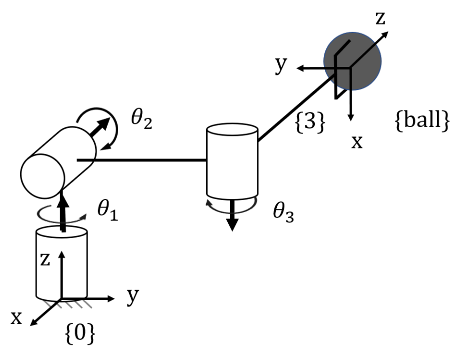


Figure 4: Robot configuration when ball is released.

The spatial body velocity $\mathcal{V}_b$ at $t = 0$ can be computed by calculating the body Jacobian at the given configuration:

$$\mathcal{V}_b(0) = J_b(\theta(0))\dot{\theta}(0) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

At release, the ball moves with constant angular velocity $\omega_b = (0, 0, 1)$ and constant linear velocity $v_b = (1, 0, 1)$ (both ω_b and v_b are expressed with respect to the end-effector frame at $t = 0$. Denoting by $T_{3,ball}(t) \in SE(3)$ the position and orientation of the ball at time t with respect to the end-effector frame at $t = 0$, we have

$$T_{3,ball}(t) = \begin{bmatrix} e^{[\omega_b]t} & v_b t \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos t & -\sin t & 0 & t \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$T_{0,ball}(t)$ is then

$$\begin{aligned} T_{0,ball}(t) &= T_{03}(0)T_{3,ball}(t) = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t & 0 & t \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -1 & -1-t \\ -\sin t & -\cos t & 0 & 1 \\ -\cos t & \sin t & 0 & 1-t \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Note that we use $T_{03}(0)$ here because $T_{3,ball}(t)$ describes the displacement of the ball with respect to frame $\{3\}$ at $t = 0$.

Problem 4 (30 points)

Figure 5(a) shows the Staubli 6R robot, with its zero configuration shown in Figure 5(b).

(a) Draw appropriate link reference frames, and derive the Denavit-Hartenberg parameters for the first three joints only.

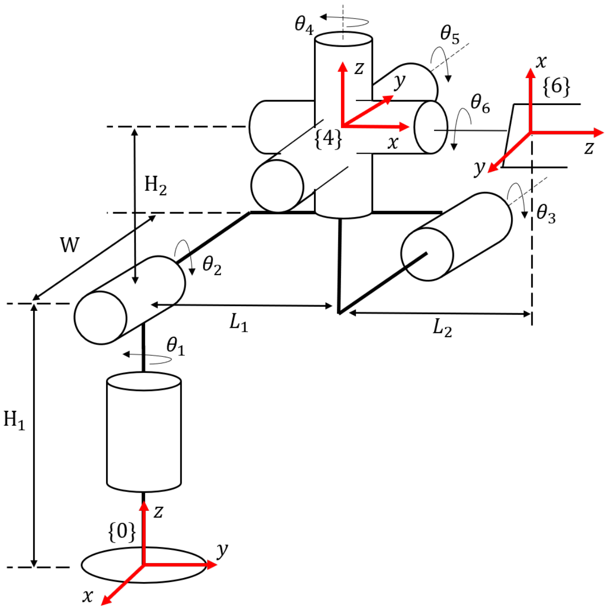
(b) The forward kinematics can be expressed in the form

$$T_{06}(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} M_4 e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M_6.$$

Find M_4, M_6, and the $\mathcal{S}_i$, $i = 1, \ldots, 6$.

(c) Letting $T_{46} = e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M_6$, find $\dot{T}_{46} T_{46}^{-1}$.

(a) 6R open chain robot, tx40.

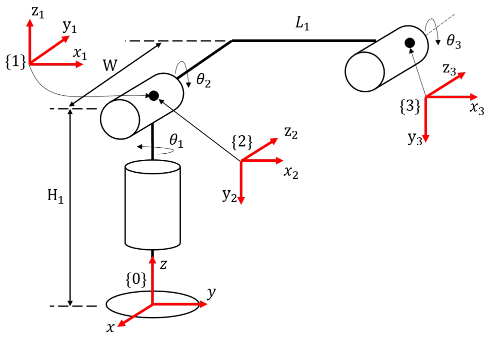


(b) Frame assignments in zero configuration.

Figure 5: Figure for Problem 4

Solution

(a) One possible set of link frames is shown in Figure 6(a). The corresponding Denavit-Hartenberg parameters are shown in Figure 6(b).



(a)

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	0	H_1	$90° + \theta_1$
2	$-90°$	0	0	θ_2
3	0	L_1	W	θ_3

(b)

Figure 6: Link frames and corresponding Denavit-Hartenberg parameters.

(b) Write $T_{06} = T_{04}T_{46}$ with $T_{04} = e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}e^{[\mathcal{S}_3]\theta_3}e^{[\mathcal{S}_4]\theta_4}M_4$ and $T_{46} = e^{[\mathcal{S}_5]\theta_5}e^{[\mathcal{S}_6]\theta_6}M_6$, where M_4 is the {4} frame expressed in the {0} frame and M_6 is the {6} frame expressed in the {4} frame, both at the zero configuration. Then

$$M_4 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & L_1 \\ 0 & 0 & 1 & H_1 + H_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_6 = \begin{bmatrix} 0 & 0 & 1 & L_2 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The corresponding joint screws are obtained from Figures 7(b) and 7(d) as follows:

$$\begin{aligned}
\mathcal{S}_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T \\
\mathcal{S}_2 &= \begin{pmatrix} -1 & 0 & 0 & 0 & -H_1 & 0 \end{pmatrix}^T \\
\mathcal{S}_3 &= \begin{pmatrix} -1 & 0 & 0 & 0 & -H_1 & L_1 \end{pmatrix}^T \\
\mathcal{S}_4 &= \begin{pmatrix} 0 & 0 & 1 & L_1 & 0 & 0 \end{pmatrix}^T \\
\mathcal{S}_5 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^T \\
\mathcal{S}_6 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T
\end{aligned}$$

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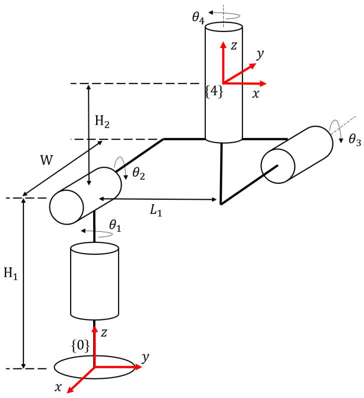
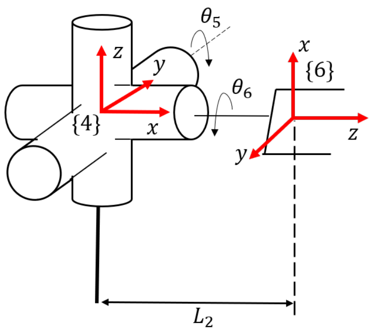


i	ω_i	q_i	v_i
1	(0,0,1)	(0,0,0)	(0,0,0)
2	(-1,0,0)	$(0,0,H_1)$	$(0,-H_1,0)$
3	(-1,0,0)	$(-W,L_1,H_1)$	$(0,-H_1,L_1)$
4	(0,0,1)	$(0,L_1,H_1+H_2)$	$(L_1,0,0)$

(b)

(a)



i	ω_i	q_i	v_i
5	(0,1,0)	(0,0,0)	(0,0,0)
6	(1,0,0)	(0,0,0)	(0,0,0)

(d)

(c)

Figure 7: Figure for Problem 4(b).

(c) Given $T_{46} = e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M_6$,

$$\begin{aligned} \dot{T}_{46} &= [\mathcal{S}_5] e^{[\mathcal{S}_5]\theta_5} \dot{\theta}_5 e^{[\mathcal{S}_6]\theta_6} M_6 + e^{[\mathcal{S}_5]\theta_5} [\mathcal{S}_6] e^{[\mathcal{S}_6]} \dot{\theta}_6 M_6 \\ T_{46}^{-1} &= M_6^{-1} e^{-[\mathcal{S}_6]\theta_6} e^{-[\mathcal{S}_5]\theta_5} \end{aligned}$$

Therefore

$$\dot{T}_{46} T_{46}^{-1} = [\mathcal{S}_5]\dot{\theta}_5 + e^{[\mathcal{S}_5]\theta_5} [\mathcal{S}_6] e^{-[\mathcal{S}_5]\theta_5} \dot{\theta}_6$$

Letting $\dot{T}_{46} T_{46}^{-1} = [\mathcal{V}_{46}]$,

$$\mathcal{V}_{46} = \mathcal{S}_5 \dot{\theta}_5 + \mathrm{Ad}_{e^{[\mathcal{S}_5]\theta_5}}(\mathcal{S}_6)\dot{\theta}_6$$

Using the results obtained in (b),

$$\begin{aligned} \mathcal{V}_{46} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^T \dot{\theta}_5 + \begin{pmatrix} \cos\theta_5 & 0 & -\sin\theta_5 & 0 & 0 & 0 \end{pmatrix}^T \dot{\theta}_6 \\ &= \begin{pmatrix} \dot{\theta}_6 \cos\theta_5 & \dot{\theta}_5 & -\dot{\theta}_6 \sin\theta_5 & 0 & 0 & 0 \end{pmatrix}^T, \end{aligned}$$

or

$$\dot{T}_{46} T_{46}^{-1} = \begin{bmatrix} 0 & \dot{\theta}_6 \sin\theta_5 & \dot{\theta}_5 & 0 \\ -\dot{\theta}_6 \sin\theta_5 & 0 & -\dot{\theta}_6 \cos\theta_5 & 0 \\ -\dot{\theta}_5 & \dot{\theta}_6 \cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Final Examination
6-9 PM, June 5, 2020
CLOSED BOOK, CLOSED NOTES

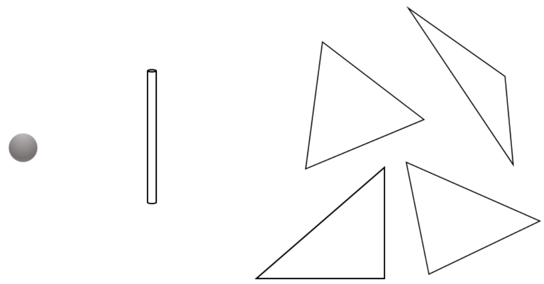
Problem 1 (30 points)

A popular children's magnetic construction kit consists of magnetic balls, legs of various lengths, and triangles of various shapes and sizes (see Figure 1(a)).

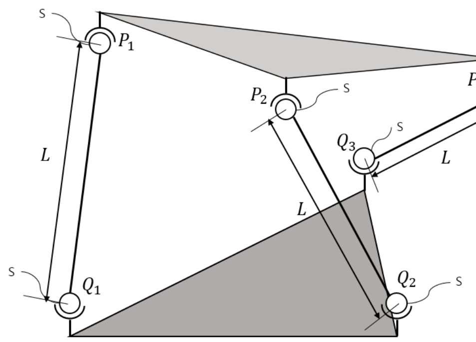
(a) The platform constructed in Figure 1(b) can be modelled as a $3 \times SS$ platform, where S denotes a spherical (or ball-in-socket) joint. Assume that the base platform is fixed to ground. Use Grübler's formula to find the degrees of freedom of this platform. Explain in words and pictures each of the degrees of freedom.

(b) Derive a set of equations that describe the configuration space of the general platform of part (a). Use a counting argument to find the degrees of freedom of this platform. Does your answer agree with your result from Grübler's formula obtained in part (a)? Explain your answer. (*Hint*: Consider using the coordinates of the three corners P_1, P_2, P_3 of the moving platform as the variables.)

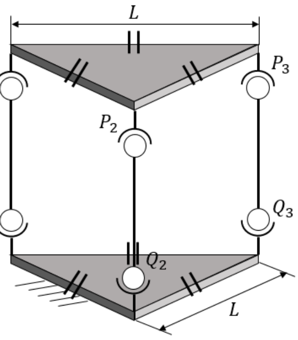
(c) Now consider the symmetric platform of Figure 1(b), in which the legs are all of length 1, and the platforms are all equilateral triangles of length 1. Does the symmetric platform have the same degrees of freedom as the general plaform, or are there additional degrees of freedom introduced? Explain your answer.



(a) Parts



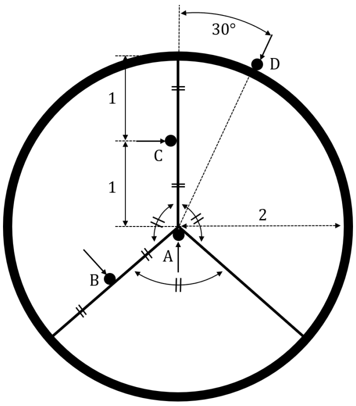
(b) General $3 \times SS$ platform



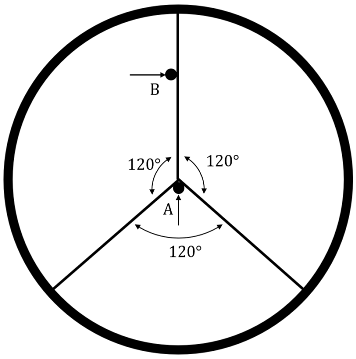
(c) Symmetric $3 \times SS$ platform

Figure 1: Figures for Problem 1

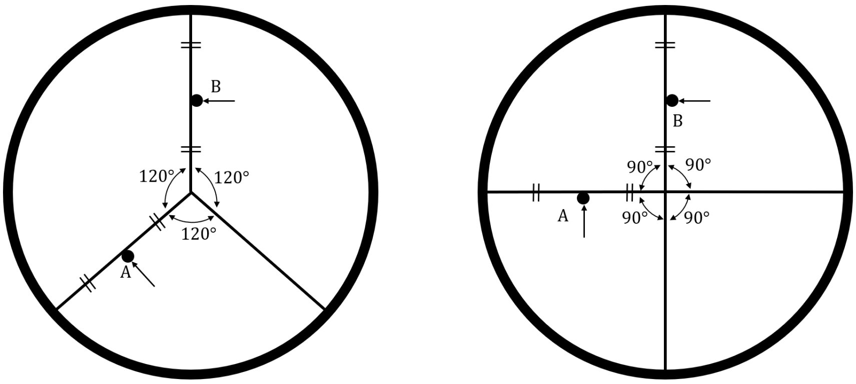
Problem 2 (30 points)
A robot with a four-fingered hand is learning how to steer a car.
(a) Suppose the robot uses four fingers to grasp a three-spoked symmetric steering wheel as shown in Figure 2(a). Assuming the four contacts are frictionless point contacts, determine if this grasp is force closure. (As a reminder, $\sin 30° = 1/2$ and $\cos 30° = \sqrt{3}/2$.)
(b) The robot now uses only two fingers to grasp the steering wheel as shown in Figure 2(b). Assume point contact A is frictionless while point contact B has friction coefficient $\mu = 1$. Suppose contact B can be moved to anywhere on the wheel. Draw all possible locations for B such that the grasp is force closure. In your drawing be sure to clearly indicate whether these locations are on the inside or outside of the wheel, or which side of the spokes they are on.
(c) The designer of the steering wheel must decide how many spokes are needed for safe driving. The robot uses exactly two fingers to grasp the midpoint of two spokes that are adjacent to each other (see Figure 2(c) for an illustration of a wheel with three spokes and four spokes, respectively). The spokes must be spaced symmetrically about the center of the wheel. Assuming that the friction coefficient $\mu = 1$ at both contacts, what is the minimum number of spokes that guarantees a two-fingered force closure grasp?



(a) A three-spoked wheel



(b) Two-finger grasp



(c) Minimum number of spokes

Figure 2: Steering wheels for Problem 2

Problem 3 (30 points)

DJI Robomasters is a popular robot competition in which teams of wheeled mobile robots and drones cooperate to shoot down moving targets. Referring to Figure 3, let {s} be the fixed frame, {r} be a frame attached to the wheeled mobile robot, {c} be a frame attached to the robot cannon, {d} be a frame attached to the drone, and {t} be a frame attached to the target.

(a) Let $R_{sr}(t) \in SO(3)$ and $p_{sr}(t) \in \mathbb{R}^3$ be the orientation and position of the robot frame {r} with respect to the fixed frame {s}. The robot is being escorted by a drone that rotates at 1 rad/sec about the z-axis of the robot {r} frame. Suppose the orientation $R_{sd}(0)$ and position $p_{sd}(0)$ of the drone at time $t = 0$ are known. Find $R_{sd}(t)$ and $p_{sd}(t)$ as a function of time t.

(b) Now suppose that the drone also slowly rotates about its own {d} frame z-axis at a rate of -1 rad/sec while circling the robot as explained in part (a). Suppose the orientation $R_{sd}(0)$ and position $p_{sd}(0)$ of the drone at time $t = 0$ are known. Find $R_{sd}(t)$ and $p_{sd}(t)$ as a function of time t.

(c) The robot now uses its cannon to fire a ball at the target frame {t}. Let {b} be a frame attached to the ball, and let the motion of {b} be given by

$$T_{cb} = \begin{bmatrix} I & p_{cb} \\ 0 & 1 \end{bmatrix} \in SE(3), \quad p_{cb} = (0, 0, t), \ t \geq 0.$$

Suppose the target is described by the ellipsoid $x^2+y^2+4z^2 \leq 1$ in frame {t} coordinates. Determine whether the ball hits the target when the motions of the cannon and target frame are given by

$$T_{sc} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{st} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \sqrt{2}t \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{\sqrt{2}}t + 3\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

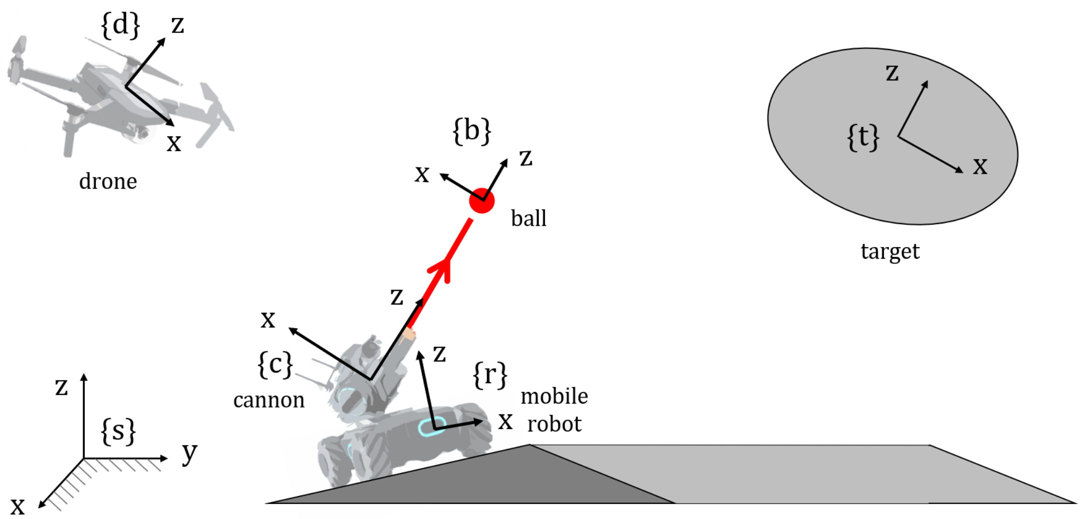


Figure 3: Figure for Problem 1.

Problem 4 (30 points)

The Cubli is a cube-shaped robot that can move by flipping over its edges and vertexes. Suppose a Cubli is placed on a 5×5 grid as shown in Figure 4(a). Each edge of the Cubli is of length L, and a frame {b} is attached to one of its corners. An **edge roll** is illustrated in Figure 4(b). The goal is to edge roll the cube sequentially from square ①, ②, ..., to square ⑤ as shown in Figure 4(c). Let $T_{sb}(n)$ denote the rigid body transformation from the fixed frame {s} to the cube frame {b} when the cube is placed in square ⓝ. Observe that

$$T_{sb}(1) = \begin{bmatrix} 0 & 0 & -1 & -L \\ 0 & -1 & 0 & 2L \\ -1 & 0 & 0 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Each $T_{sb}(n)$ can be expressed as follows:

$$\begin{aligned} T_{sb}(2) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}} \\ T_{sb}(3) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}}e^{[\mathcal{S}_2]\frac{\pi}{2}} \\ T_{sb}(4) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}}e^{[\mathcal{S}_2]\frac{\pi}{2}}e^{[\mathcal{S}_3]\frac{\pi}{2}} \\ T_{sb}(5) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}}e^{[\mathcal{S}_2]\frac{\pi}{2}}e^{[\mathcal{S}_3]\frac{\pi}{2}}e^{[\mathcal{S}_4]\frac{\pi}{2}}. \end{aligned}$$

Calculate the screws $\mathcal{S}_k \in \mathbb{R}^6$, $k = 1, 2, 3, 4$.

(b) Now consider a $4R$ robot with forward kinematics

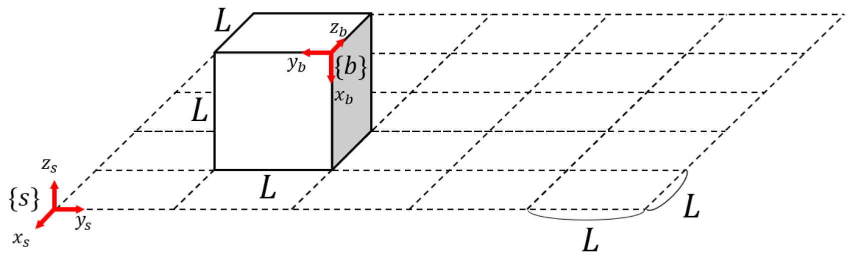
$$T = Me^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}e^{[\mathcal{S}_3]\theta_3}e^{[\mathcal{S}_4]\theta_4},$$

where the $\mathcal{S}_i$ are the screws that you obtained in part (a) and M is set to the identity I. For this robot, assign appropriate base, end-effector, and link reference frames, and derive the Denavit-Hartenberg parameters. (If you were unable to obtain the $\mathcal{S}_i$ in part (a), then choose arbitrary values for the $\mathcal{S}_i$ and solve this problem.)

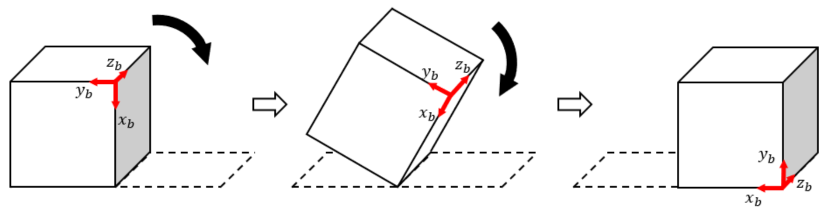
(c) The Cubli can also perform a **vertex roll** as shown in Figure 4(d). A vertex roll is equivalent to two edge rolls, i.e.,

$$\begin{aligned} T_{sb}(3) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}}e^{[\mathcal{S}_2]\frac{\pi}{2}} \\ &= T_{sb}(1)e^{[\mathcal{S}]\theta} \end{aligned}$$

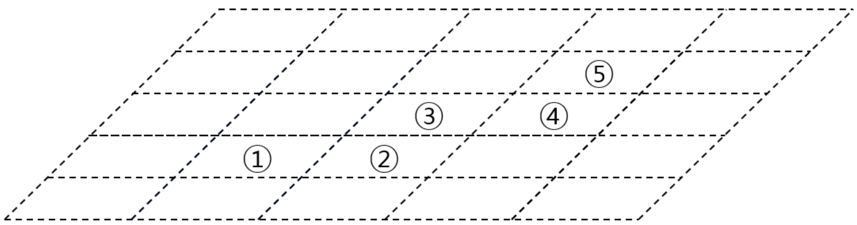
for some screw $\mathcal{S} = (\omega, v)$ and angle θ. Find the screw axis direction vector ω and the angle θ. If you cannot find explicit values for ω and v, then explain how to find these using the exponential and logarithm formulas on $SO(3)$ and $SE(3)$.



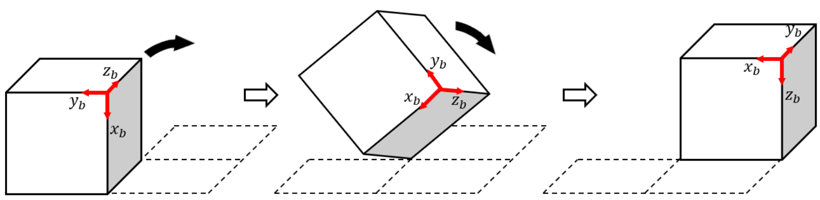
(a)



(b)



(c)



(d)

Figure 4

Problem 5 (30 points)

Figure 5 shows a soccer robot in its zero configuration. The dimensions and joint variables are as indicated in the figure.

(a) Assuming the robot's kicking leg is at the initial configuration $\theta = (\theta_1, \theta_2, \theta_3) = (-\pi/2, -\pi/2, 0)$, find the velocity vector of the foot tip with respect to the center-of-mass (COM) frame when $\dot{\theta} = (2, 1, 2)$ (in units of rad/sec).

(b) Assume $W = 1$ and $L = \sqrt{2}$. When the foot strikes the ball at configuration $\theta = (0, 0, 0)$, find the joint torque vector $\tau = (\tau_1, \tau_2, \tau_3)$ so that the impact force is only in the y_c direction.

(c) Again assuming $W = 1$ and $L = \sqrt{2}$, we now wish to maximize the impact velocity v_{impact} when the foot strikes the ball. Assuming the joint velocity vector must satisfy the constraint $\|\dot{\theta}\|^2 = 1$, find $\dot{\theta}$ that maximizes $\|v_{impact}\|^2$ at the posture $\theta = (0, 0, 0)$.

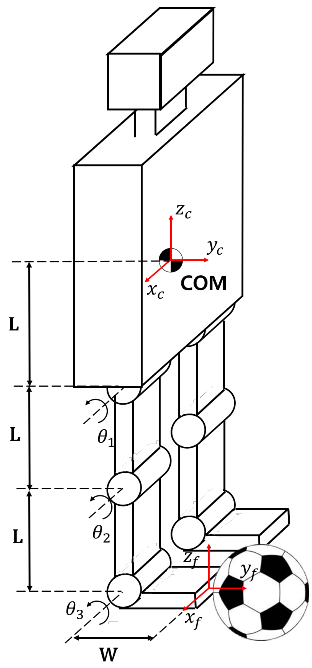


Figure 5: Soccer robot for Problem 5

Problem 6 (30 points)

(a) The 5R robot of Figure 6 (shown in its zero configuration) is used for tabletop cleaning. Given the desired end-effector frame

$$T_{sb} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & p_x \\ \sin\gamma & \cos\gamma & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where γ is an arbitrary angle and (p_x, p_y, p_z) is a point inside the robot's reachable workspace, how many inverse kinematic solutions will exist in general? (*Hint*: Observe that joint axis 5 must always be vertical.)

(b) Suppose the robot is at the posture $\theta = \{\pi/4, \pi/4, -\pi/2, \pi/4, 0\}$. Derive the constraints on the joint velocity vector $\dot{\theta}$ so that the end-effector always maintains contact with the tabletop.

(c) Suppose we want to resize links L_1 and L_2 to improve the cleaning capability of the robot. Assume $L_1 + L_2 =$ constant, ignore joints 4 and 5, and set $L_3 = 0$. Assuming the robot operates over the region defined by $0 < \theta_1, \theta_2, \theta_3 < \pi/2$, from the perspective of positioning manipulability (you may ignore the end-effector orientation for this problem), is it better if $L_1 > L_2$ or $L_1 < L_2$? (If you cannot make progress on this problem for general $(\theta_1, \theta_2, \theta_3)$, then consider the problem for the specific configuration $(\theta_1, \theta_2, \theta_3) = (0, 60°, 30°)$.

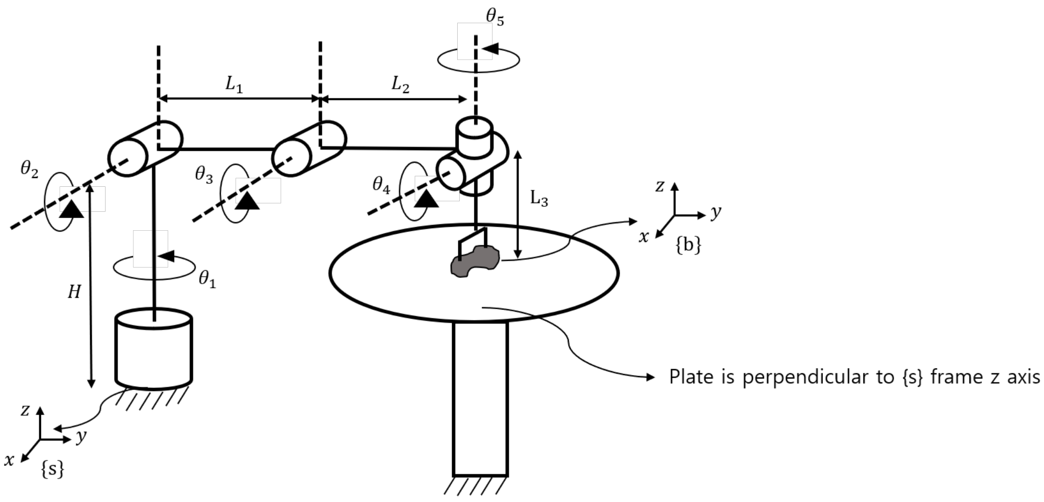


Figure 6: $5R$ robot for Problem 6

M2794.0027 Introduction to Robotics
Final Examination Solutions
6-9 PM, June 5, 2020
CLOSED BOOK, CLOSED NOTES

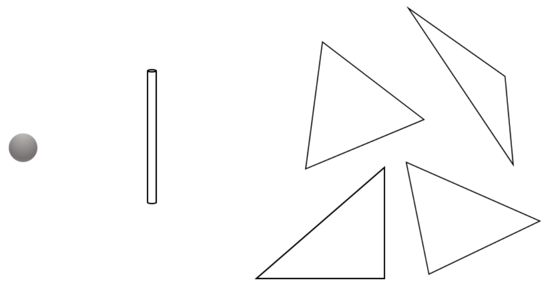
Problem 1 (30 points)

A popular children's magnetic construction kit consists of magnetic balls, legs of various lengths, and triangles of various shapes and sizes (see Figure 2(a)).

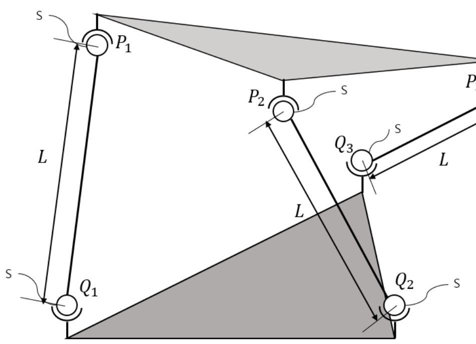
(a) The platform constructed in Figure 2(b) can be modelled as a $3 \times SS$ platform, where S denotes a spherical (or ball-in-socket) joint. Assume that the base platform is fixed to ground. Use Grübler's formula to find the degrees of freedom of this platform. Explain in words and pictures each of the degrees of freedom.

(b) Derive a set of equations that describe the configuration space of the general platform of part (a). Use a counting argument to find the degrees of freedom of this platform. Does your answer agree with your result from Grübler's formula obtained in part (a)? Explain your answer. (*Hint*: Consider using the coordinates of the three corners P_1, P_2, P_3 of the moving platform as the variables.)

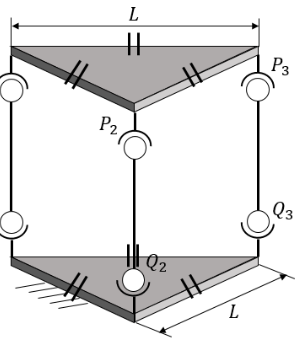
(c) Now consider the symmetric platform of Figure 2(b), in which the legs are all of length 1, and the platforms are all equilateral triangles of length 1. Does the symmetric platform have the same degrees of freedom as the general plaform, or are there additional degrees of freedom introduced? Explain your answer.



(a) Parts



(b) General $3 \times SS$ platform



(c) Symmetric $3 \times SS$ platform

Figure 1: Figures for Problem 1

Solution

(a) The number of links (including the ground link) is $N = 5$. The number of joints is $J = 6$, with each $f_i = 3$, $i = 1, \ldots, J$ for the ball-and-socket joints. From the spatial version of Grübler's formula dof $= 6(N - 1 - J) + \sum_{i=1}^{J} f_i$, it can be determined that this mechanism has six degrees of freedom:

1. Each leg has a torsional degree of freedom as shown in Figure 2(a), acccounting for three degrees of freedom.

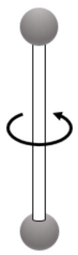
2. The moving platform has three rotational degrees of freedom given by a yaw motion (Figure 2(b)) and a roll-pitch motion (Figure 2(c)).

(b) The configuration space equations can be expressed by the following six equalities:

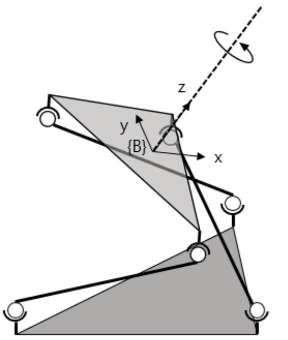
$$\begin{aligned} \|p_1 - q_1\| &= L, \quad \|p_1 - p_2\| = L \\ \|p_2 - q_2\| &= L, \quad \|p_2 - p_3\| = L \\ \|p_3 - q_3\| &= L, \quad \|p_1 - p_3\| = L \end{aligned}$$

Since the above results in six equations in nine variables (the (x, y, z) coordinates for the p_i), a dimensional counting argument implies that the platform has $9 - 6 = 3$ degrees of freedom. Taking into account the three torsional degrees of freedom of each leg (which are not captured by these equations), this result agrees with that obtained from Gruübler's formula in part (a).

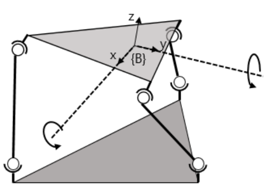
(c) Because the fixed and moving platforms are identically symmetric, the moving platform always remains parallel to the fixed platform; this is the primary difference with the general (asymmetric) case. The moving platform still possesses three rotational degrees of freedom, however; translation in the x-y plane (Figure 2(d)) and yaw motion about the z-axis (Figure 2(e)). Taking into account the three torsional degrees of freedom of each leg, the symmetric platform also has six degrees of freedom.



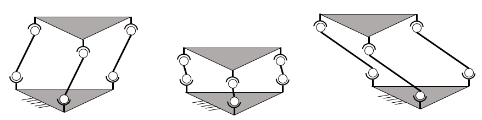
(a) Torsional degree of freedom for leg



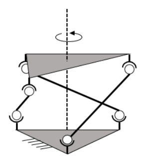
(b) Yaw motion for the general case



(c) Pitch-roll motion for the general case



(d) Tilting motion for symmetric case



(e) Yaw motion for symmetric case

Figure 2: Figures for Problem 1

Problem 2 (30 points)
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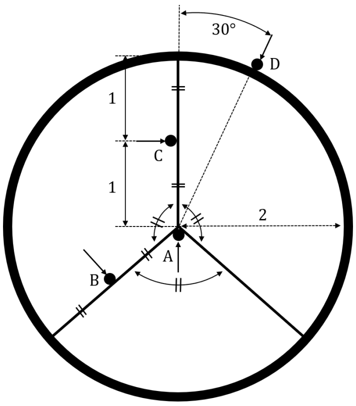
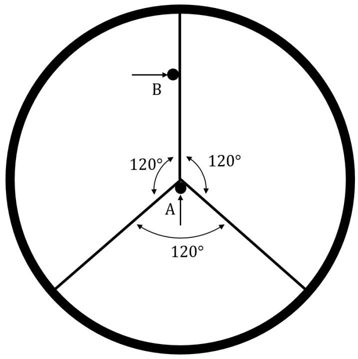
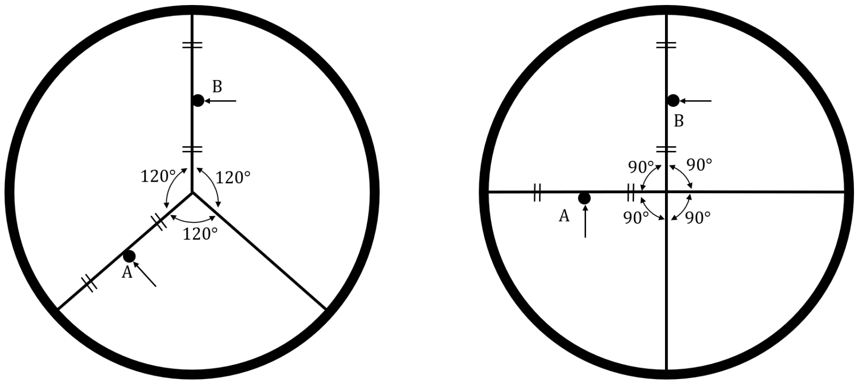
(a) Suppose the robot uses four fingers to grasp a three-spoked symmetric steering wheel as shown in Figure 3(a). Assuming the four contacts are frictionless point contacts, determine if this grasp is force closure. (As a reminder, $\sin 30° = 1/2$ and $\cos 30° = \sqrt{3}/2$.)
(b) The robot now uses only two fingers to grasp the steering wheel as shown in Figure 3(b). Assume point contact A is frictionless while point contact B has friction coefficient $\mu = 1$. Suppose contact B can be moved to anywhere on the wheel. Draw all possible locations for B such that the grasp is force closure. In your drawing be sure to clearly indicate whether these locations are on the inside or outside of the wheel, or which side of the spokes they are on.
(c) The designer of the steering wheel must decide how many spokes are needed for safe driving. The robot uses exactly two fingers to grasp the midpoint of two spokes that are adjacent to each other (see Figure 3(c) for an illustration of a wheel with three spokes and four spokes, respectively). The spokes must be spaced symmetrically about the center of the wheel. Assuming that the friction coefficient $\mu = 1$ at both contacts, what is the minimum number of spokes that guarantees a two-fingered force closure grasp?



(a) A three-spoked wheel



(b) Two-finger grasp



(c) Minimum number of spokes

Figure 3: Steering wheels for Problem 2

Solution

(a) Express the static equilibrium force closure conditions in the standard linear form $Ax = b$, where $A \in \mathbb{R}^{3\times 5}$ is given, and the objective is to determine whether a nonnegative solution $x \geq 0$ exists for any arbitrary $b \in \mathbb{R}^3$. Setting up the problem shown in Figure 4(a) in this way leads to Gauss-Jordan elimination of the following matrix:

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 & -1 \\ \sqrt{3} & \sqrt{3} & -\sqrt{3} & 0 & -\sqrt{3} \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix}.$$

Performing Gauss-Jordan elimination leads to

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 & -1 \\ \sqrt{3} & \sqrt{3} & -\sqrt{3} & 0 & -\sqrt{3} \\ 0 & 0 & 2 & -1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix} = [I \quad S]$$

Since there exists some $w \in \mathbb{R}^2, w \geq 0$ such that Sw has all entries negative, force closure holds.

(b) Because the point contact at A is at a corner, it can be represented by a friction cone. From Nguyen's Theorem, force closure requires that the line connecting B and A must lie inside both friction cones. Since the friction cone at A does not contain any spokes, point contact B must be a point on the wheel inside the friction cone at A. The blue lines in Figure 4(b) indicate all possible locations for B such that the grasp is force closure.

(c) Because the friction coefficient $\mu = 1$, the angle of each friction cone is $90°$. Figure **??** shows a general wheel with n spokes and two point contacts. From Nguyen's Theorem, the blue line connecting B and A must lie inside both friction cones, or $\theta < 45°$ and $\tan\theta < 1$. From Figure 4(c),

$$\tan\theta = \frac{\sin\frac{2\pi}{n}}{1 - \cos\frac{2\pi}{n}}.$$

From the above equation, $\tan\theta = 1$ when $n = 4$, and $\tan\theta < 1$ when $n > 4$.

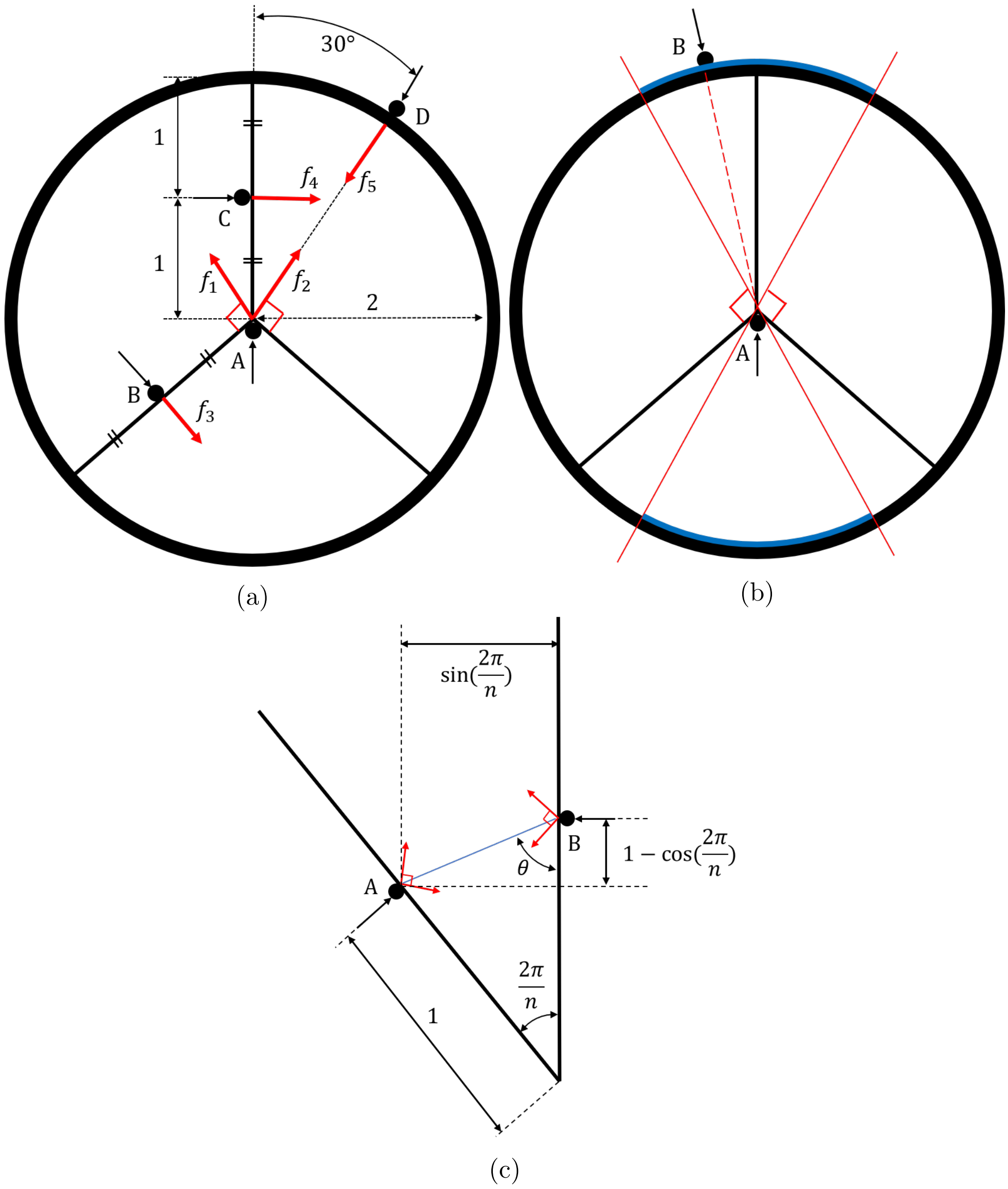


Figure 4: Figure for Problem 2

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(b) Now suppose that the drone also slowly rotates about its own {d} frame z-axis at a rate of -1 rad/sec while circling the robot as explained in part (a). Suppose the orientation $R_{sd}(0)$ and position $p_{sd}(0)$ of the drone at time $t = 0$ are known. Find $R_{sd}(t)$ and $p_{sd}(t)$ as a function of time t.
(c) The robot now uses its cannon to fire a ball at the target frame {t}. Let {b} be a frame attached to the ball, and let the motion of {b} be given by

$$T_{cb} = \begin{bmatrix} I & p_{cb} \\ 0 & 1 \end{bmatrix} \in SE(3), \quad p_{cb} = (0, 0, t), \; t \geq 0.$$

Suppose the target is described by the ellipsoid $x^2+y^2+4z^2 \leq 1$ in frame {t} coordinates. Determine whether the ball hits the target when the motions of the cannon and target frame are given by

$$T_{sc} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{st} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \sqrt{2}t \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{\sqrt{2}}t + 3\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

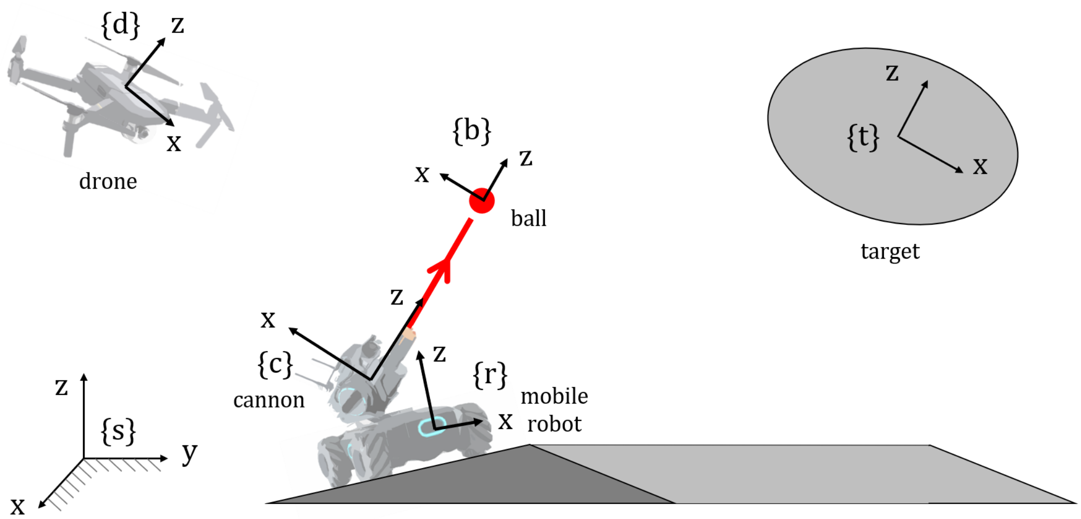


Figure 5: Figure for Problem 1.

Solution

(a) The drone at time $t = 0$ seen from the mobile robot {r} is $T_{r_0 d_0} = T_{sr_0}^{-1} T_{sd_0}$. At time t, the drone is rotated around the z-axis of the robot by t radians:

$$T_{rd}(t) = \text{Rot}(\hat{z}, t) T_{r_0 d_0}.$$

At time t, the drone seen from frame {s} is thus

$$\begin{aligned} T_{sd}(t) &= T_{sr} T_{rd}(t) = T_{sr} \text{Rot}(\hat{z}, t) T_{r_0 d_0} \\ &= \begin{bmatrix} R_{sr}(t) & p_{sr}(t) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{Rot}(\hat{z}, t) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sr}(0) & p_{sr}(0) \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} R_{sd}(0) & p_{sd}(0) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_{sr}(t)\text{Rot}(\hat{z}, t) R_{sr}(0)^T R_{sd}(0) & R_{sr}(t)\text{Rot}(\hat{z}, t) R_{sr}(0)^T (p_{sd}(0) - p_{sr}(0)) + p_{sr}(t) \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

$$\begin{aligned} \therefore R_{sd}(t) &= R_{sr}(t)\text{Rot}(\hat{z}, t) R_{sr}(0)^T R_{sd}(0) \\ p_{sd}(t) &= R_{sr}(t)\text{Rot}(\hat{z}, t) R_{sr}(0)^T (p_{sd}(0) - p_{sr}(0)) + p_{sr}(t). \end{aligned}$$

(b) The motion of the drone, this time with an additional rotation about the z-axis of the drone frame {d}, can be obtained by multiplying a rotation matrix to $T_{sd}(t)$ obtained in (a) as follows:

$$T_{sd}(t) = T_{sr} \text{Rot}(\hat{z}, t) T_{r_0 d_0} \text{Rot}(\hat{z}, -t).$$

$$\begin{aligned} \therefore R_{sd}(t) &= R_{sr}(t)\text{Rot}(\hat{z}, t) R_{sr}(0)^T R_{sd}(0) \text{Rot}(\hat{z}, -t) \\ p_{sd}(t) &= R_{sr}(t)\text{Rot}(\hat{z}, t) R_{sr}(0)^T (p_{sd}(0) - p_{sr}(0)) + p_{sr}(t). \end{aligned}$$

(c) The ball seen from frame {s} is $T_{sb} = T_{sc} T_{cb}$. The ball seen from the target frame {t} is

$$\begin{aligned} T_{tb} &= T_{st}^{-1} T_{sb} = T_{st}^{-1} T_{sc} T_{cb} \\ &= \begin{bmatrix} R_{st}^T & -R_{st}^T p_{st} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sc} & p_{sc} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & p_{cb} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} R_{st}^T R_{sc} & R_{st}^T (R_{sc} p_{cb} + p_{sc} - p_{st}) \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Therefore, the position of the ball seen from the target frame {t} at time t is

$$p_{tb}(t) = R_{st}^T (R_{sc} p_{cb} + p_{sc} - p_{st}) = (-\frac{1}{2}t, \frac{3}{2}t, -2\sqrt{2})^T.$$

Substituting this into the equation for the ellipsoid,

$$\begin{gathered} (-\tfrac{1}{2}t)^2 + (\tfrac{3}{2}t)^2 + 4(-2\sqrt{2})^2 \leq 1 \\ \tfrac{5}{2}t^2 \leq -31. \end{gathered}$$

Since the above equation does not have any real-value solutions, the ball misses the target.

Problem 4 (30 points)

The Cubli is a cube-shaped robot that can move by flipping over its edges and vertexes. Suppose a Cubli is placed on a 5×5 grid as shown in Figure 6(a). Each edge of the Cubli is of length L, and a frame {b} is attached to one of its corners. An **edge roll** is illustrated in Figure 6(b). The goal is to edge roll the cube sequentially from square ①, ②, ..., to square ⑤ as shown in Figure 6(c). Let $T_{sb}(n)$ denote the rigid body transformation from the fixed frame {s} to the cube frame {b} when the cube is placed in square ⓝ. Observe that

$$T_{sb}(1) = \begin{bmatrix} 0 & 0 & -1 & -L \\ 0 & -1 & 0 & 2L \\ -1 & 0 & 0 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Each $T_{sb}(n)$ can be expressed as follows:

$$\begin{aligned} T_{sb}(2) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}} \\ T_{sb}(3) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}}e^{[\mathcal{S}_2]\frac{\pi}{2}} \\ T_{sb}(4) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}}e^{[\mathcal{S}_2]\frac{\pi}{2}}e^{[\mathcal{S}_3]\frac{\pi}{2}} \\ T_{sb}(5) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}}e^{[\mathcal{S}_2]\frac{\pi}{2}}e^{[\mathcal{S}_3]\frac{\pi}{2}}e^{[\mathcal{S}_4]\frac{\pi}{2}}. \end{aligned}$$

Calculate the screws $\mathcal{S}_k \in \mathbb{R}^6$, $k = 1, 2, 3, 4$.

(b) Now consider a $4R$ robot with forward kinematics

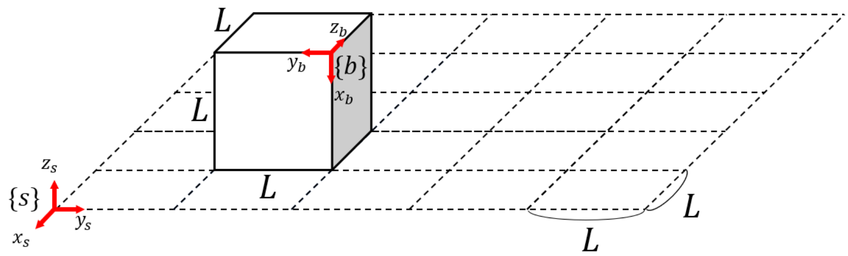
$$T = Me^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}e^{[\mathcal{S}_3]\theta_3}e^{[\mathcal{S}_4]\theta_4},$$

where the $\mathcal{S}_i$ are the screws that you obtained in part (a) and M is set to the identity I. For this robot, assign appropriate base, end-effector, and link reference frames, and derive the Denavit-Hartenberg parameters. (If you were unable to obtain the $\mathcal{S}_i$ in part (a), then choose arbitrary values for the $\mathcal{S}_i$ and solve this problem.)

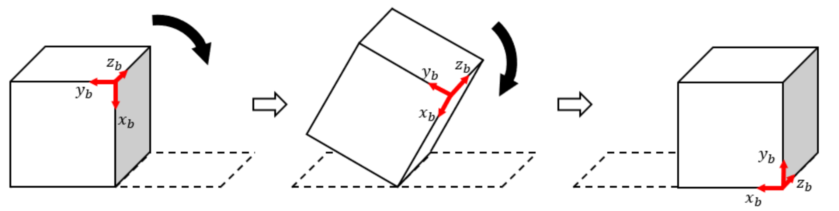
(c) The Cubli can also perform a **vertex roll** as shown in Figure 6(d). A vertex roll is equivalent to two edge rolls, i.e.,

$$\begin{aligned} T_{sb}(3) &= T_{sb}(1)e^{[\mathcal{S}_1]\frac{\pi}{2}}e^{[\mathcal{S}_2]\frac{\pi}{2}} \\ &= T_{sb}(1)e^{[\mathcal{S}]\theta} \end{aligned}$$

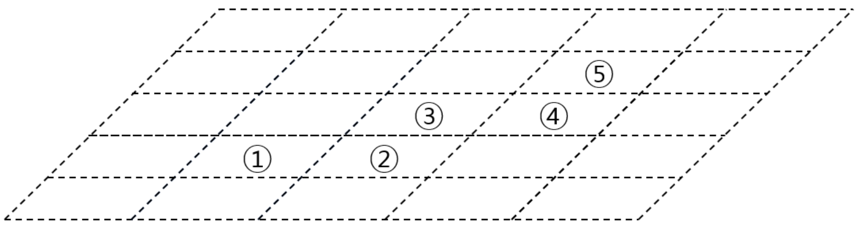
for some screw $\mathcal{S} = (\omega, v)$ and angle θ. Find the screw axis direction vector ω and the angle θ. If you cannot find explicit values for ω and v, then explain how to find these using the exponential and logarithm formulas on $SO(3)$ and $SE(3)$.



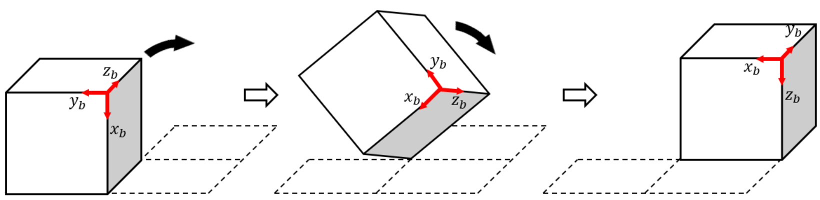
(a)



(b)



(c)



(d)

Figure 6

Solution

(a) Referring to Figure 7,

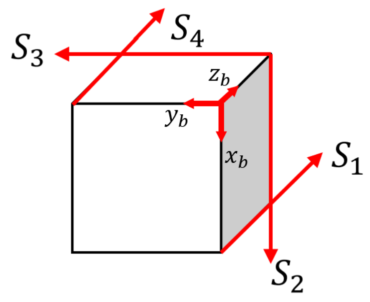


Figure 7

We can calculate body screws with the Fig 7.

$$\begin{aligned}
\mathcal{S}_1 &= \begin{bmatrix} 0 & 0 & 1 & 0 & -L & 0 \end{bmatrix}^T \\
\mathcal{S}_2 &= \begin{bmatrix} 1 & 0 & 0 & 0 & L & 0 \end{bmatrix}^T \\
\mathcal{S}_3 &= \begin{bmatrix} 0 & 1 & 0 & -L & 0 & 0 \end{bmatrix}^T \\
\mathcal{S}_4 &= \begin{bmatrix} 0 & 0 & 1 & L & 0 & 0 \end{bmatrix}^T
\end{aligned}$$

(b) The corresponding robot is shown in Figure 8.

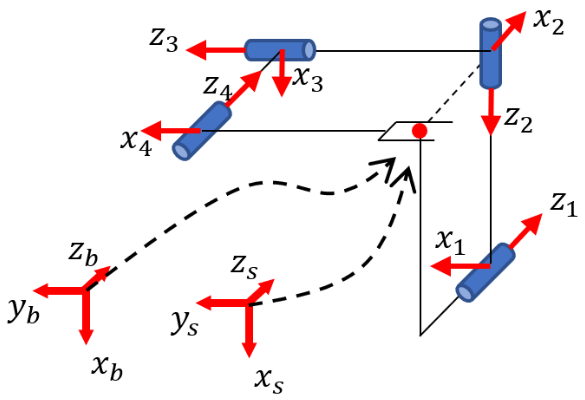


Figure 8

The D-H parameters for the link frames assigned in Figure 8 are as follows:

i	α_{i-1}	a_{i-1}	d_i	ϕ_i
1	0	L	L	$\theta_1 + \pi/2$
2	$\pi/2$	0	$-L$	$\theta_2 + \pi/2$
3	$\pi/2$	0	L	$\theta_3 + \pi/2$
4	$\pi/2$	0	$-L$	$\theta_4 + \pi/2$
5	0	$-L$	0	$-\pi/2$

(Note: Full credit was given even if you used a different set of screws, as long as the frames were properly assigned and the correct D-H parameters obtained).
(c) Define the transformation T as follows:

$$e^{[\mathcal{S}_1]\frac{\pi}{2}}e^{[\mathcal{S}_2]\frac{\pi}{2}} = T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$$

with $\mathcal{S}_1, \mathcal{S}_2$ as obtained in (a), so that the rotation R can be extracted from T from the formulas

$$\begin{aligned} \mathrm{tr} R &= 1 + 2\cos\theta \\ [\omega] &= \frac{1}{2\sin\theta}(R - R^T). \end{aligned}$$

From the above formulas we get $\omega = \frac{1}{\sqrt{3}}(1,1,1)$ and $\theta = \frac{2\pi}{3}$. Alternatively, both θ and ω can be obtained from the following intuitive argument: Referring to Figure 6(d), the frame rotates as shown Figure 9(a) with a single vertex roll, in which the axes are shifted clockwise. Continuing the vertex roll, Figure 9(b) illustrates a sequence of three vertex rolls, at which the frame orientation returns to the original orientation. From this one can infer that θ is $\frac{2\pi}{3}$ and $\omega = \frac{1}{\sqrt{3}}(1,1,1)$.

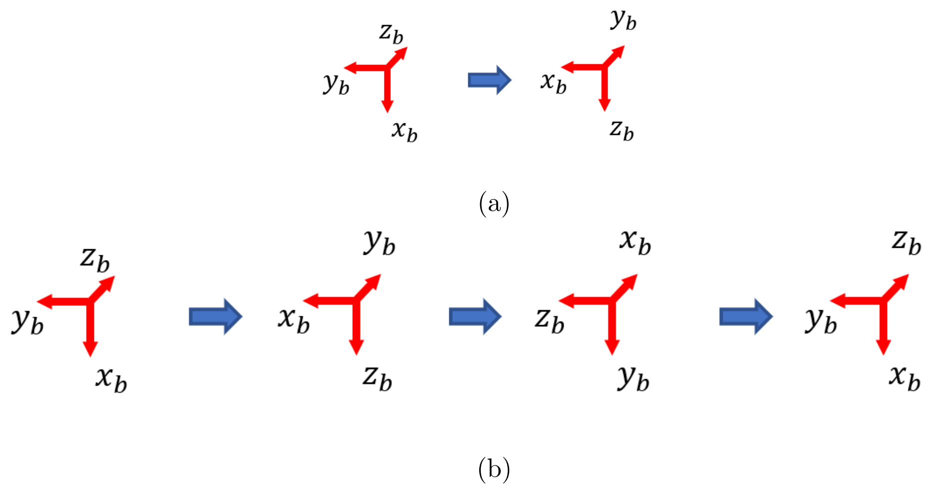


Figure 9

Problem 5 (30 points)
Figure 5 shows a soccer robot in its zero configuration. The dimensions and joint variables are as indicated in the figure.
(a) Assuming the robot's kicking leg is at the initial configuration $\theta = (\theta_1, \theta_2, \theta_3) = (-\pi/2, -\pi/2, 0)$, find the velocity vector of the foot tip with respect to the center-of-mass (COM) frame when $\dot{\theta} = (2, 1, 2)$ (in units of rad/sec).
(b) Assume $W = 1$ and $L = \sqrt{2}$. When the foot strikes the ball at configuration $\theta = (0, 0, 0)$, find the joint torque vector $\tau = (\tau_1, \tau_2, \tau_3)$ so that the impact force is only in the y_c direction.
(c) Again assuming $W = 1$ and $L = \sqrt{2}$, we now wish to maximize the impact velocity v_{impact} when the foot strikes the ball. Assuming the joint velocity vector must satisfy the constraint $\|\dot{\theta}\|^2 = 1$, find $\dot{\theta}$ that maximizes $\|v_{impact}\|^2$ at the posture $\theta = (0, 0, 0)$.

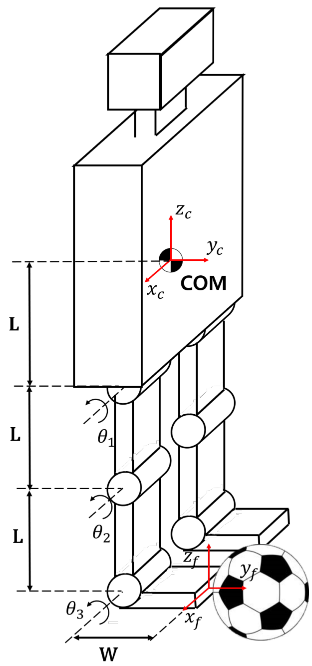


Figure 10: Soccer robot for Problem 5

Solution

(a) The initial configuration is shown in Figure 11(a) (links are shown in bold). To find the velocity of the foot tip with respect to the COM frame, first calculate the body Jacobian and the velocity with respect to foot tip frame as follows:

i	$\hat{\omega}_i$	q_i	v_i
1	$(1,0,0)$	$(0,-W-L,L)$	$(0,L,W+L)$
2	$(1,0,0)$	$(0,-W,L)$	$(0,L,W)$
3	$(1,0,0)$	$(0,-W,0)$	$(0,0,W)$

$$\Rightarrow \quad J_b = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ L & L & 0 \\ W+L & W & W \end{bmatrix}$$

$$\mathcal{V}_b = J_b\dot{\theta} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 3L \\ 2L+5W \end{bmatrix} = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}.$$

Since ω_b and v_b are respectively the angular and linear velocities of the foot tip expressed in the foot tip frame, the corresponding angular and linear velocities of the foot tip frame expressed in COM frame can be obtained via multiplication by R_{cf}. From Figure 11(a),

$$R_{cf} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Finally, the velocity vector of the foot tip with respect to the COM frame is

$$v_c = R_{cf}v_b = \begin{bmatrix} 0 \\ -3L \\ -2L-5W \end{bmatrix}$$

(b) The configuration at which the robot kicks the ball is shown in Figure 11(b). In order for the robot to kick the ball only in the y_c direction, the following wrench needs to be generated at the foot tip:

$$\mathcal{F}_f = (0,0,0,0,f_y,0)^T$$

Letting J_f be the body Jacobian for the foot tip frame, the required joint torque $\tau = (\tau_1,\tau_2,\tau_3)^T$ is then obtained as $\tau = {J_f}^T\mathcal{F}_f$:

i	$\hat{\omega}_i$	q_i	v_i
1	$(1,0,0)$	$(0,-1,2\sqrt{2})$	$(0,2\sqrt{2},1)$
2	$(1,0,0)$	$(0,-1,\sqrt{2})$	$(0,\sqrt{2},1)$
3	$(1,0,0)$	$(0,-1,0)$	$(0,0,1)$

$$\Rightarrow \quad J_f = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\tau = {J_f}^T\mathcal{F}_f = f_c(2\sqrt{2},\sqrt{2},0)^T$$

(c) The optimization problem can be stated as

$$\max_{\dot{\theta}} \|v_{impact}\|^2 \text{ subject to } \|\dot{\theta}\|^2 = 1.$$

From the result obtained in (b), the spatial velocity of the foot tip expressed in the foot tip frame is

$$\mathcal{V}_f = J_f \dot{\theta} = (\omega_f, v_f) \Rightarrow v_f = \begin{bmatrix} 0 & 0 & 0 \\ 2\sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \dot{\theta}.$$

The impact velocity of the foot tip expressed in the COM frame can be obtained via multiplication by R_{cf}:

$$v_{impact} = R_{cf} \begin{bmatrix} 0 & 0 & 0 \\ 2\sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \dot{\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 2\sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \dot{\theta} = \begin{bmatrix} 0 & 0 & 0 \\ 2\sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix} \dot{\theta}.$$

Letting $A \triangleq \begin{bmatrix} 0 & 0 & 0 \\ 2\sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & 1 \end{bmatrix}$, the optimization problem can be restated as

$$\max_{\dot{\theta}} \|A\dot{\theta}\|^2 \text{ subject to } \|\dot{\theta}\|^2 = 1.$$

The first-order necessary conditions are then

$$\begin{aligned} \dot{\theta}^T \dot{\theta} - 1 &= 0 \\ 2A^T A \dot{\theta} - 2\lambda \dot{\theta} &= 0, \end{aligned}$$

i.e., $A^T A \dot{\theta} = \lambda \dot{\theta}$. Therefore the $\dot{\theta}$ that maximizes $\|v_{impact}\|^2$ is the eigenvector corresponding to the largest eigenvalue of $A^T A$:

$$A^T A = \begin{bmatrix} 9 & 5 & 1 \\ 5 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow \det(A^T A - sI) = \begin{vmatrix} 9-s & 5 & 1 \\ 5 & 3-s & 1 \\ 1 & 1 & 1-s \end{vmatrix} = -s(s-12)(s-1)$$

From the above characteristic polynomial, the largest eigenvalue is 12 and corresponding eigenvector is $(7,4,1)^T$. Since $\|\dot{\theta}\|^2 = 1$, the answer is $\dot{\theta} = \frac{1}{\sqrt{66}}(7,4,1)^T$.

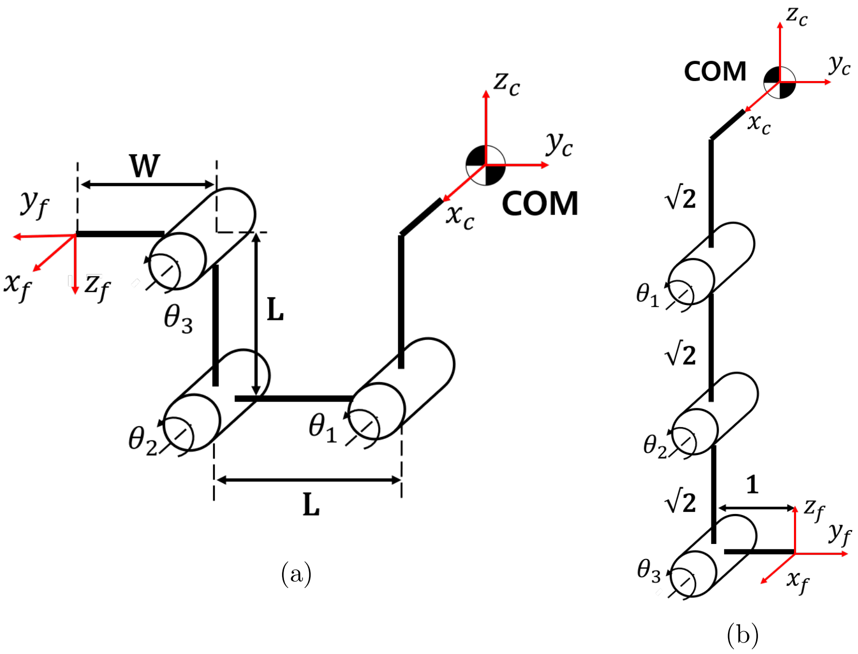


Figure 11: Figure for Problem 5

Problem 6 (30 points)
(a) The 5R robot of Figure 6 (shown in its zero configuration) is used for tabletop cleaning. Given the desired end-effector frame

$$T_{sb} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & p_x \\ \sin\gamma & \cos\gamma & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where γ is an arbitrary angle and (p_x, p_y, p_z) is a point inside the robot's reachable workspace, how many inverse kinematic solutions will exist in general? (*Hint*: Observe that joint axis 5 must always be vertical.)
(b) Suppose the robot is at the posture $\theta = \{\pi/4, \pi/4, -\pi/2, \pi/4, 0\}$. Derive the constraints on the joint velocity vector $\dot{\theta}$ so that the end-effector always maintains contact with the tabletop.
(c) Suppose we want to resize links L_1 and L_2 to improve the cleaning capability of the robot. Assume $L_1 + L_2 =$ constant, ignore joints 4 and 5, and set $L_3 = 0$. Assuming the robot operates over the region defined by $0 < \theta_1, \theta_2, \theta_3 < \pi/2$, from the perspective of positioning manipulability (you may ignore the end-effector orientation for this problem), is it better if $L_1 > L_2$ or $L_1 < L_2$? (If you cannot make progress on this problem for general $(\theta_1, \theta_2, \theta_3)$, then consider the problem for the specific configuration $(\theta_1, \theta_2, \theta_3) = (0, 60°, 30°)$.

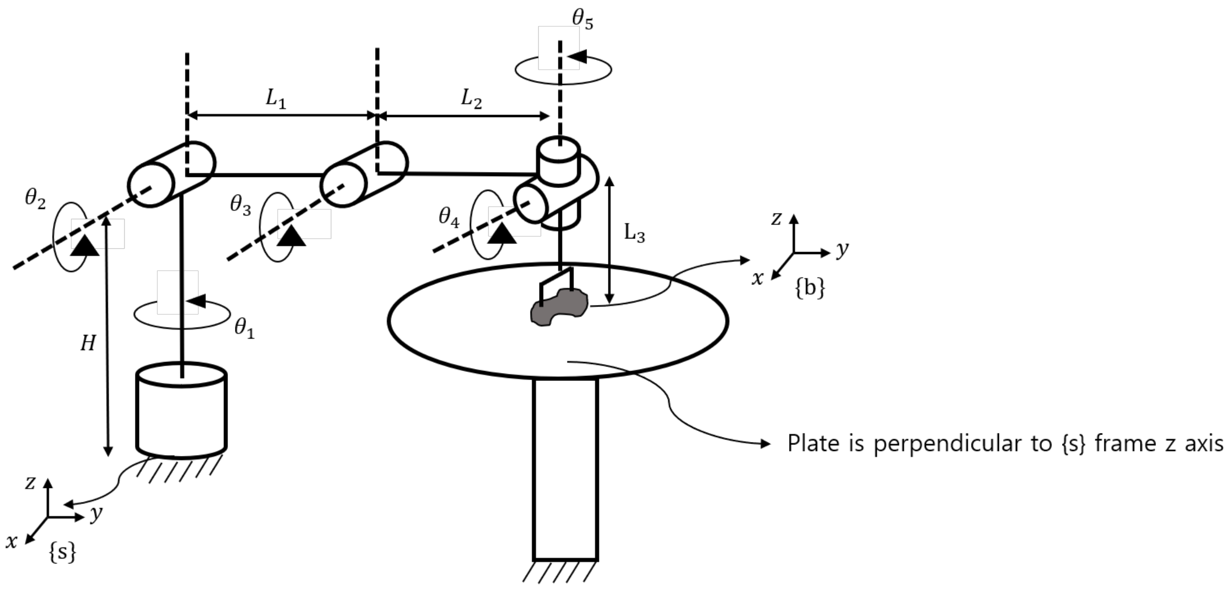


Figure 12: $5R$ robot for Problem 6

Solution

(a) First match R_{sb} of the desired end-effector frame. Since the z-axis of the end-effector frame is the same as that of the $\{s\}$ frame, $\theta_2+\theta_3+\theta_4$ is $2n\pi$ for integer n. To match angle γ in the desired end-effector frame, $\theta_1+\theta_5=\gamma$.

Next, we match p_{sb} of the desired end-effector frame using $(\theta_1,\theta_2,\theta_3)$. There are four inverse kinematic solutions for the $3R$ arm to reach desired the p_{sb}, i.e., the usual left-arm/right-arm, elbow-up/elbow-down postures.The number of possible solutions $(\theta_1,\theta_2,\theta_3)$ is four, and for each solution θ_4 and θ_5 can be uniquely determined from the earlier constraint equations. The total number of general inverse kinematic solutions is therefore 4.

(b) At the given posture, the body Jacobian matrix can be calculated as follows:

	ω	q	v
1	$(0,0,1)$	$(0, -\frac{L_2+L_1}{\sqrt{2}}, 0)$	$(-\frac{L_2+L_1}{\sqrt{2}}, 0, 0)$
2	$(1,0,0)$	$(0, -\frac{L_2+L_1}{\sqrt{2}}, L_3+\frac{L_2-L_1}{\sqrt{2}})$	$(0, L_3+\frac{L_2-L_1}{\sqrt{2}}, \frac{L_2+L_1}{\sqrt{2}})$
3	$(1,0,0)$	$(0, -\frac{L_2}{\sqrt{2}}, L_3+\frac{L_2}{\sqrt{2}})$	$(0, L_3+\frac{L_2}{\sqrt{2}}, \frac{L_2}{\sqrt{2}})$
4	$(1,0,0)$	$(0,0, L_3)$	$(0, L_3, 0)$
5	$(0,0,1)$	$(0,0, 0)$	$(0,0,0)$

$$J_b(\theta) = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ -\frac{L_1+L_2}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & L_3 + \frac{L_2-L_1}{\sqrt{2}} & L_3 + \frac{L_2}{\sqrt{2}} & L_3 & 0 \\ 0 & \frac{L_2+L_1}{\sqrt{2}} & \frac{L_2}{\sqrt{2}} & 0 & 0 \end{bmatrix},$$

The body velocity is then given by $V_b = J_b(\theta)\dot{\theta}$, and for the end-effector to maintain contact with the tabletop, the z-axis comonent of V_b must be 0. Therefore we have $\frac{L_2+L_1}{\sqrt{2}}\dot{\theta}_2 + \frac{L_2}{\sqrt{2}}\dot{\theta}_3 = 0$. (c) The body Jacobian matrix can be calculated as follows:

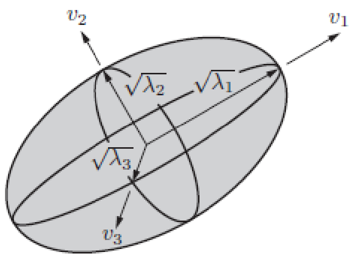
	ω	q	v
1	$(0, s_{23}, c_{23})$	$(0, -L_2 - L_1c_3, L_1s_3)$	$(-L_2c_{23} - L_1c_2, 0, 0)$
2	$(1, 0, 0)$	$(0, -L_2 - L_1c_3, L_1s_3)$	$(0, L_1s_3, L_2 + L_1c_3)$
3	$(1, 0, 0)$	$(0, -L_2, 0)$	$(0, 0, L_2)$

$$J_b(\theta) = \begin{bmatrix} 0 & 1 & 1 \\ s_{23} & 0 & 0 \\ c_{23} & 0 & 0 \\ -L_2c_{23} - L_1c_2 & 0 & 0 \\ 0 & L_1s_3 & 0 \\ 0 & L_2 + L_1c_3 & L_2 \end{bmatrix},$$

Since we only have to consider the positioning manipulability, only the lower 3×3 submatrix needs to be considered; we refer to this submatrix by J_{b1}. Denoting the end-effector linear velocity by v we have $v = J_{b1}\dot{\theta}$. Assuming J_{b1} is invertible, the unit joint velocity condition can be expressed as

$$1 = \dot{\theta}^T\dot{\theta} = (J_{b1}^{-1}v)^T(J_{b1}^{-1}v) = v^T(J_{b1}J_{b1}^T)^{-1}v.$$

The v satisfying the above equation corresponds to the manipulability ellipsoid with principal axes whose lengths are $\sqrt{\lambda_i}$, where λ_i are the eigenvalues of $J_{b1}J_{b1}^T$.



The volume V of the ellipsoid is proportional to the product of the semi-axis lengths, i.e., V is proportional to $\sqrt{\lambda_1\lambda_2\lambda_3} = \sqrt{\det(J_{b1}J_{b1}^T)}$., where

$$J_{b1}J_{b1}^T = \begin{bmatrix} (L_2c_{23} + L_1c_2)^2 & 0 & 0 \\ 0 & L_1^2s_3^2 & L_1s_3(L_2 + L_1c_3) \\ 0 & L_1s_3(L_2 + L_1c_3) & L_2^2 + (L_2 + L_1c_3)^2 \end{bmatrix},$$

$$det(J_{b1}J_{b1}^T) = (L_1c_2 + L_2c_{23})^2 L_1^2 s_3^2 L_2^2$$

Since $0 < \theta_2, \theta_3 < \pi/2$, it follows that $\cos\theta_2 > \cos(\theta_2 + \theta_3)$. Therefore it is better to have $L_1 > L_2$ from the perspective of manipulability.