M2794.002700 Introduction to Robotics
Midterm Examination 1
April 18, 2019
CLOSED BOOK, CLOSED NOTES

Problem 1 (40 points)
(a) The 4R mechanism of Figure 1(a) is used to draw ellipsoids, in which a pen is attached to the intersection point between link AD and link BC. Let (x_a, y_a) be the coordinates for point A and (x_b, y_b) be the coordinates for point B. Use Grübler's formula to find the degrees of freedom of the mechanism. Derive a set of constraint equations in terms of (x_a, y_a, x_b, y_b) to support your answer.
(b) Carefully explain why the pen traces an ellipsoid. (*Hint*: Referring to Figure 1(b), recall that an ellipsoid is defined by its two focal points P and Q; the sum of the lengths of PS and SQ is always constant for any point S on the ellipsoid, i.e. $d_1 + d_2 = constant$.)
(c) The mechanism of Figure 1(a) is now used to construct the mechanism shown in Figure 1(c). Use Grübler's formula to find the degrees of freedom of the mechanism of Figure 1(c). Does your result agree with physical intuition? Explain your answer.

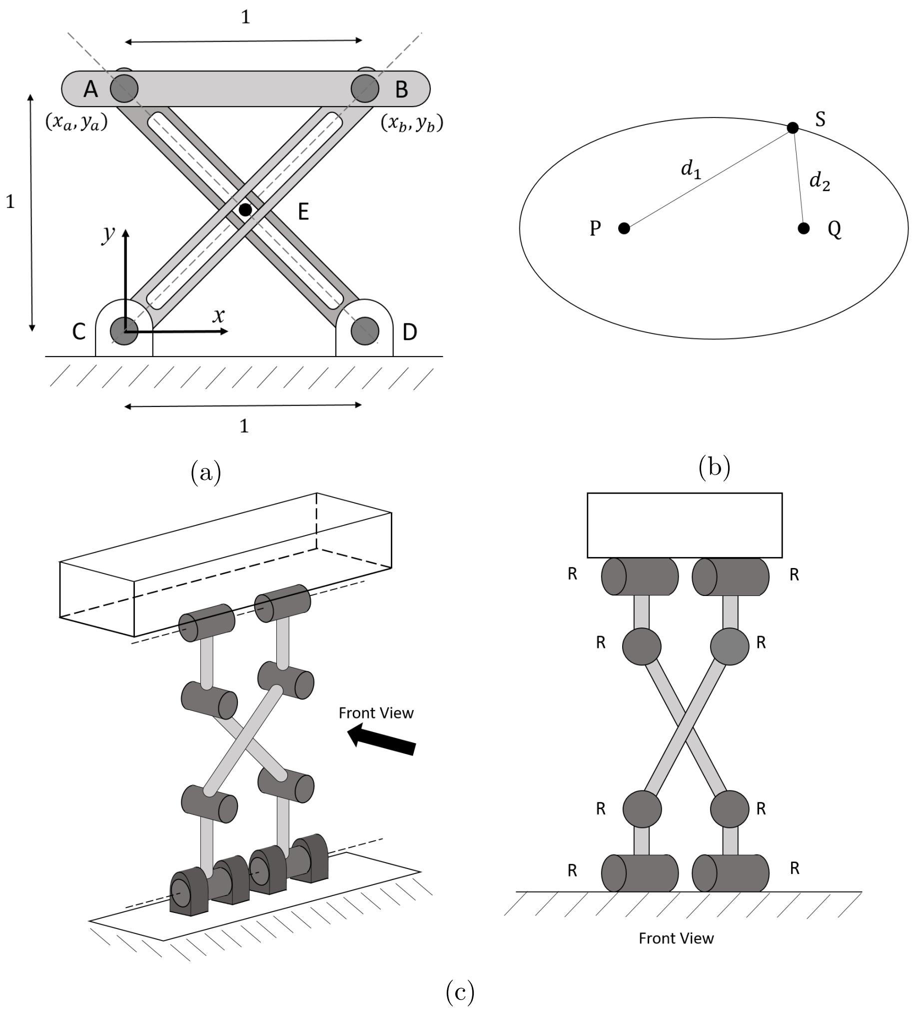


Figure 1: Mechanisms for Problem 1

Problem 2 (40 points)

(a) Four frictionless point contacts **a**, **b**, **c**, **d** are used to fix the mouse shown in Figure 2(a). The two ends of the mouse are semicircles of radius $r = 4$. Contacts **b**, **c**, **d** are fixed as shown, while the location of contact **a** is to be determined. For what values of θ_a is the mouse in force closure? (you may assume $\frac{\pi}{4} \le \theta_a \le \frac{5\pi}{4}$)

(b) Now assume point contact **a** is located at $\theta_a = \frac{\pi}{2}$, and show that this grasp is equivalent to two point contacts with friction located at the intersection of the normal lines of contact (**e** and **f**), with the normal lines of contact defining the friction cones (see Figure 2(b)).

(c) One point contact **a** with friction coefficient $\mu_a = 0.5$ and one frictionless cylinder **b** of radius r_b are used to fix the boomerang as shown in Figure 2(c). Find the range of r_b so that the boomerang is in force closure. You may use the result of Problem 2(b).

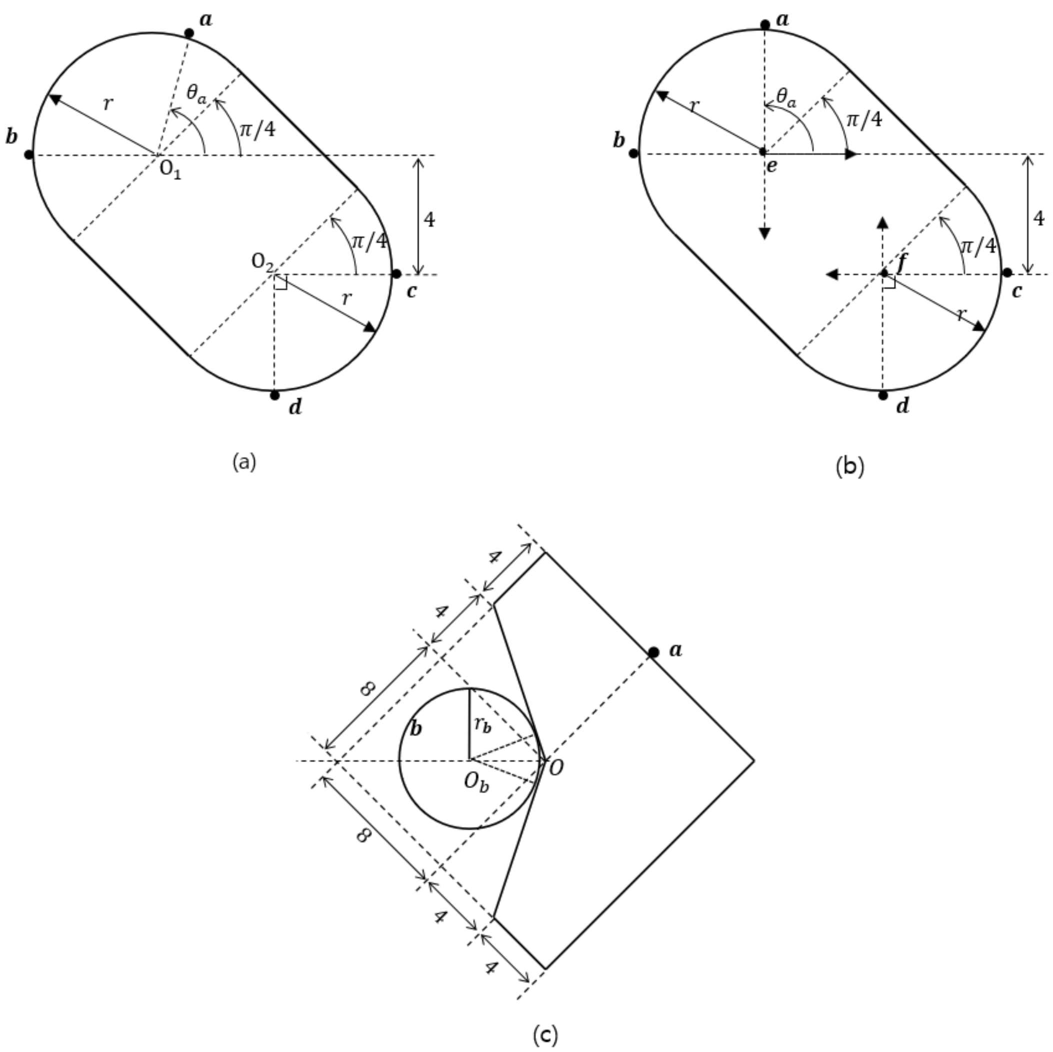


Figure 2: (a) a mouse, (b) a mouse with imaginary holes, (c) a boomerang

Problem 3 (40 points)

(a) Referring to Figure 3(a), $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$ are three points on a rigid body. Let $q_1, q_2, q_3 \in \mathbb{R}^3$ be the coordinates for these points in the initial configuration, and $q_1', q_2', q_3' \in \mathbb{R}^3$ be the coordinates for these points in the final configuration. Then q_i and q_i' are related by

$$q_i' = Rq_i + p, \quad i = 1, 2, 3.$$

for some rotation matrix $R \in SO(3)$ and vector $p \in \mathbb{R}^3$. Assuming R is known, prove the following:

$$(q_1' - q_3') \times (q_2' - q_3') = R((q_1 - q_3) \times (q_2 - q_3)).$$

(b) A new event for the world drone racing championships has been introduced, in which a pilot must land a drone on a moving platform (see Figure 3(b)). References frames are attached and labelled as shown. Let

$$T_{ij} = \begin{bmatrix} R_{ij} & p_{ij} \\ 0 & 1 \end{bmatrix} \in SE(3)$$

be the rigid body transformation matrix describing the position and orientation of frame $\{j\}$ as seen from frame $\{i\}$, with

$$T_{0p} = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 5 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The landing platform rotates at $\omega = 1$ rad/sec. Assuming that $t = 0$ at the instant shown in the figure, with $R_{01}(0) = R_{02}(0) = I$, find $T_{p2}(t)$ as a function of t.

(c) Suppose $p_{30}(t), p_{32}(t), p_{3p}(t)$ can be measured using the drone camera. Using the $T_{0p}(t)$ you obtained in part (b), find $R_{p3}(t)$. Is your $R_{p3}(t)$ unique? If not, describe any additional conditions needed to obtain a unique $R_{p3}(t)$. (*Hint*: The results of (a) may be helpful.)

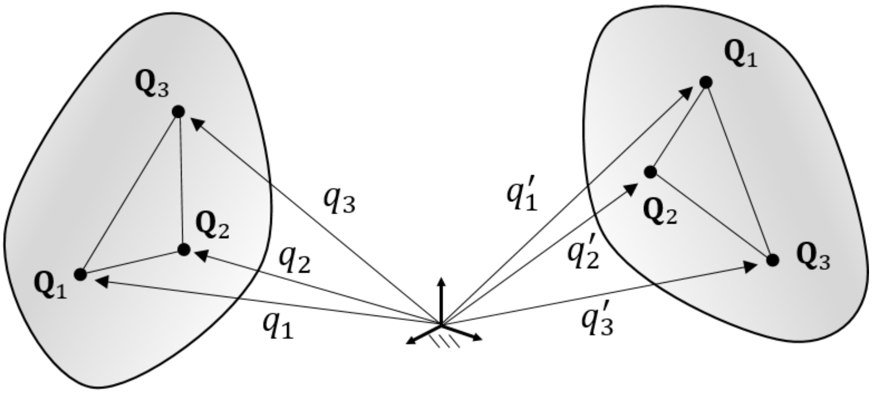
(d) The camera is attached to the drone via a three-axis gimbal. Attach frame $\{4\}$ to the drone and frame $\{5\}$ to the camera, and let

$$R_{45} = \mathrm{Rot}(\hat{\omega}_1, \alpha) \cdot \mathrm{Rot}(\hat{\omega}_2, \beta) \cdot \mathrm{Rot}(\hat{\omega}_3, \gamma),$$

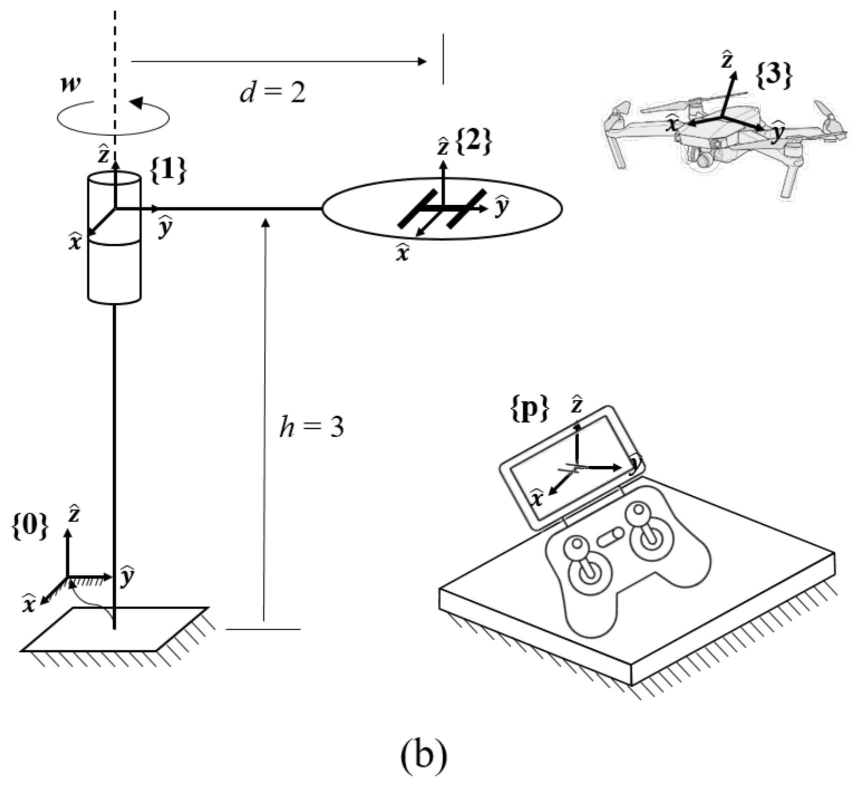
with $\hat{\omega}_1 = (0, 0, 1)$, $\hat{\omega}_2 = (0, 1, 0)$, and $\hat{\omega}_3 = (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. Suppose R_{04} is of the form

$$R_{04} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \quad r_{13} = 1$$

Find, if it exists, angles (α, β, γ) in the range $0 < \alpha < 2\pi$, $0 \le \beta \le \pi$, $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$, such that $R_{05} = I$.



(a)

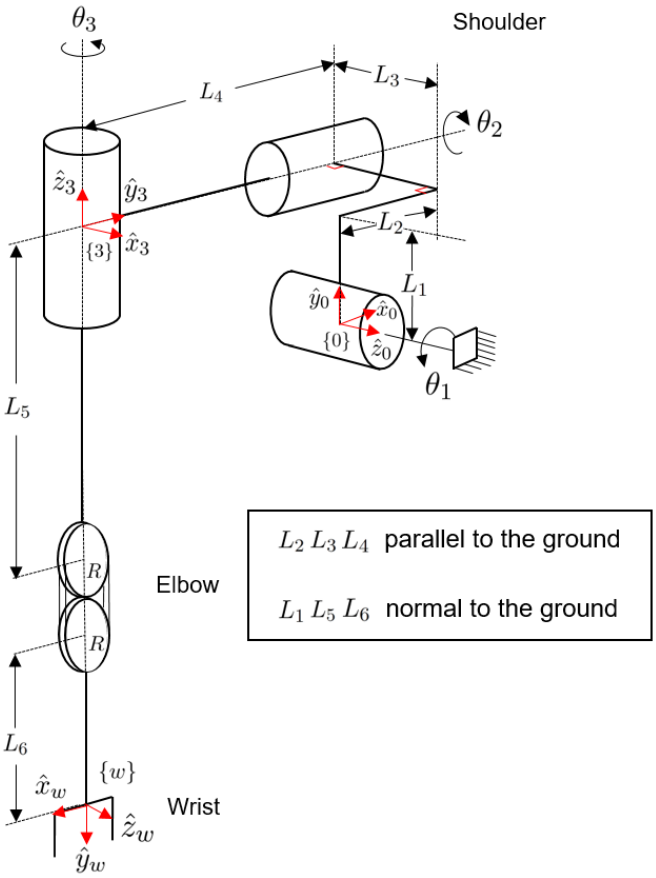


(b)

Figure 3: Figures for Problem 3.

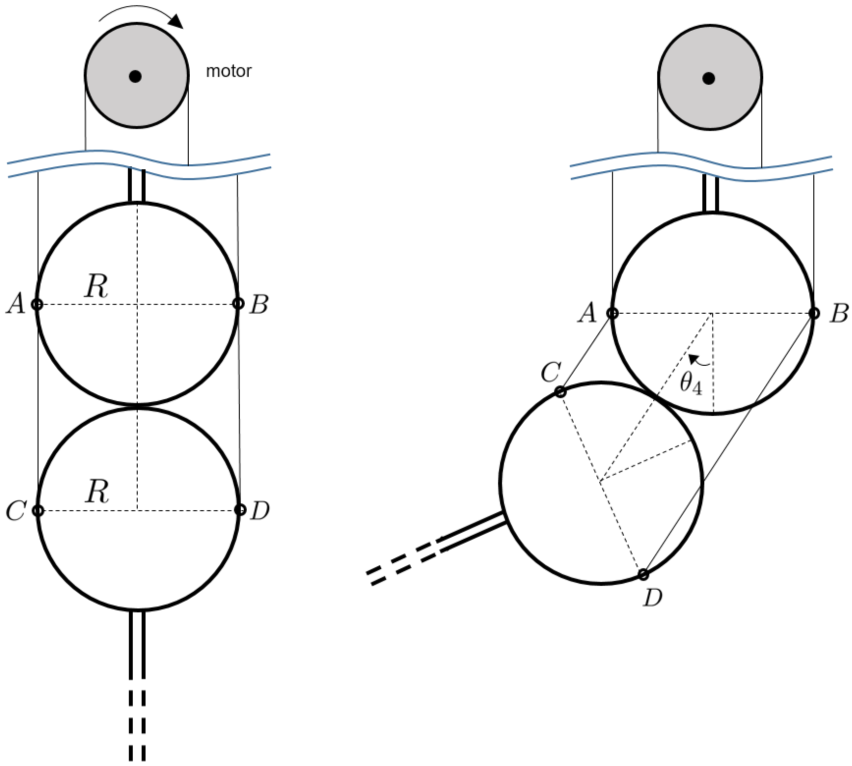
Problem 4 (40 points)
In this problem we analyze the forward kinematics of the Ambidex tendon-driven arm shown in Figure 4(a).
(a) **Shoulder kinematics**: Referring to Figure 4(b), draw link frames {1} and {2}, and find all the Denavit-Hartenberg parameters needed to evaluate T_{03}.
(b) **Elbow kinematics**: Referring to Figure 5(a), the elbow mechanism consists of two circular discs of radius R, connected to a motor by a tendon. The motor actuates the tendon in such a way that $\overline{AC} + \overline{BD}$ is always constant, and the two circular discs roll against each other without slip. Derive T_{3w} as a function of θ_4.
(c) **Wrist kinematics**: The two-DoF wrist mechanism consists of two spheres of radius r, connected to a motor by a set of tendons. The two spheres roll against each other without slip. The tendons (l_3, l_4) and (l_5, l_6) are actuated by the same motor, so that $l_3 + l_4 = l_5 + l_6$ is always constant. The point of contact point between the two spheres is expressed in spherical coordinates (ϕ, ψ) as shown in Figure 5(b). Derive $T_{ww'}$ in terms of φ and ψ. You may express your answer using Rot(·) and Trans(·) notation.


(a) Ambidex arm developed by Naver Labs.

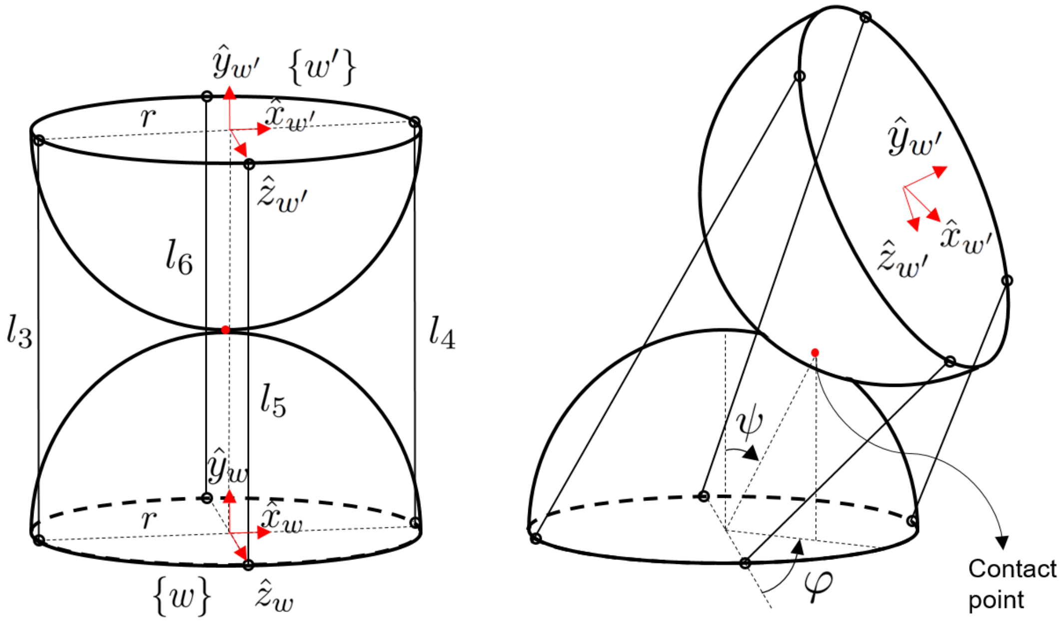


(b) Frame assignments in zero configuration.

Figure 4: Ambidex mechanisms.



(a) Elbow mechanism.



(b) Wrist mechanism.

Figure 5: Ambidex mechanisms.

M2794.0027 Introduction to Robotics
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Problem 1

(a) Applying the planar version of Grübler's formula to the mechanism, with the number of links $N = 4$ (3 links and ground) and the number of joints $J = 4$ (four revolute joints, $f_i = 1$),

$$\text{DoF} = 3(N - 1 - J) + \Sigma f_i = 3(4 - 1 - 4) + 4 = 1.$$

Now write down a set of constraint equations in terms of the coordinates (x_a, y_a, x_b, y_b). Noting that $\overline{AB}$ is of unit length while $\overline{AD}, \overline{BC}$ are of length $\sqrt{2}$, we have

$$\begin{aligned}(x_a - 1)^2 + {y_a}^2 &= 2 \\ {x_b}^2 + {y_b}^2 &= 2 \\ (x_a - x_b)^2 + (y_a - y_b)^2 &= 1.\end{aligned}$$

The above constitutes three constraint equations in four parameters, implying that only $4 - 3 = 1$ parameter can be chosen independently, which agrees with the previously obtained result from Grübler's formula.

(b) Referring to Figure 1, observe that the two triangles $\triangle ADB$ and $\triangle CBD$ are always congruent: $\overline{BD}$ is the common edge, while $\overline{AB} = \overline{CD} = 1$ and $\overline{AD} = \overline{CB} = \sqrt{2}$. Further observe that $\triangle AEB$ and $\triangle CED$ are also congruent: $\angle EAB = \angle ECD$ (since $\triangle ADB \equiv \triangle CBD$), $\angle EBA = \angle EDC$ (since $\angle EAB = \angle ECD$ and $\angle AEB = \angle CED$), and $\overline{AB} = \overline{CD} = 1$. It follows that $\overline{DE} = \overline{BE}$ and

$$\overline{CE} + \overline{DE} = \overline{CE} + \overline{BE} = \overline{CB} = \sqrt{2}.$$

Point E therefore traces an ellipsoid with focal points C and D.

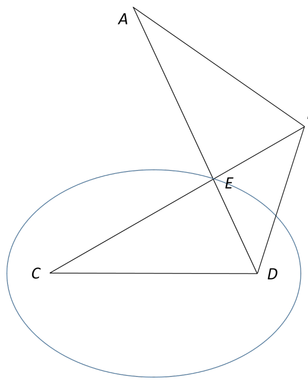


Figure 1: Ellipsoid mechanism of Problem 1

(c) Applying the spatial version of Grübler's formula leads to the following: $N = 8$ (seven links plus ground), $J = 8$ (eight revolute joints, each with $f_i = 1$), and

$$\text{DoF} = 6(N - 1 - J) + \Sigma f_i = 6(8 - 1 - 8) + 8 = 2.$$

However, from the figure it should be clear that the mechanism has three degrees of freedom: (i) The two revolute joints at the top are collinear, allowing for one dof of rotational motion; (ii) The one-dof ellipsoid mechanism of 1(a) appears in the middle, and (iii) the lower two revolute joints are also collinear, allowing for another dof of rotational motion. Applying Grübler's formula in this case therefore leads to the incorrect answer.

Problem 2

(a) Express the static equilibrium force closure conditions in the standard linear form $Ax = b$, where $A \in \mathbb{R}^{3\times 4}$ is given, and the objective is to determine whether a nonnegative solution $x \geq 0$ exists for any arbitrary $b \in \mathbb{R}^3$. Setting up the problem in this way leads to Gauss-Jordan elimination of the following matrix:

$$A = \begin{bmatrix} -\cos\theta_a & 1 & -1 & -0 \\ -\sin\theta_a & 0 & 0 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix}.$$

Reordering the columns to A' and performing Gauss-Jordan elimination leads to

$$A' = \begin{bmatrix} 1 & 0 & -1 & -\cos\theta_a \\ 0 & 1 & 0 & -\sin\theta_a \\ 0 & -4 & 4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -(\cos\theta_a + \sin\theta_a) \\ 0 & 1 & 0 & -\sin\theta_a \\ 0 & 0 & 1 & -\sin\theta_a \end{bmatrix}.$$

Force closure requires that all entries of the fourth column be negative: $-\sin\theta_a < 0$ and $-(\cos\theta_a + \sin\theta_a) < 0$, or equivalently, $0 < \theta_a < \pi$ and $-\frac{\pi}{4} < \theta_a < \frac{3\pi}{4}$. Further taking into account that $\mathbf{a}$ is restricted to lie on the semicircular arc defined by $\frac{\pi}{4} \leq \theta_a \leq \frac{5\pi}{4}$, we have

$$\frac{\pi}{4} \leq \theta_a < \frac{3\pi}{4}.$$

(b) For this part set θ_a to $\frac{\pi}{2}$, and set up the corresponding matrix A for contacts $\{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$:

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix}.$$

For the case of contacts $\{\mathbf{e}, \mathbf{f}\}$, the force closure conditions can be expressed in the form $A'x = b$, where

$$A' = \begin{bmatrix} 0 & 1 & -1 & -0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -4 & 4 \end{bmatrix}.$$

Observe that the two matrices A and A' are exactly identical.

(c) Using the results of (b), the two point contacts applied by cylinder $\mathbf{b}$ can be regarded as one friction cone located at O_b, with $\mu_{\mathbf{b}} = \tan\beta$. From Nguyen's Theorem, in order for the system to be in force closure, the line connecting $\mathbf{a}$ and $O_{\mathbf{b}}$ must lie inside both friction cones. This means that point $\mathbf{a}$ should lie inside the friction cone at $\mathbf{b}$, and $O_{\mathbf{b}}$ should lie inside the friction cone at $\mathbf{a}$. Define $d_{\mathbf{b}} = \overline{OO}_b > 0$ and $r_{\mathbf{b}} = \frac{3}{\sqrt{10}} d_{\mathbf{b}}$. Further observe that

$$\beta = \frac{\pi}{4} - \arctan\frac{4}{8} \Rightarrow \tan\beta = \frac{1 - 0.5}{1 + 1 \times 0.5} = \frac{1}{3}.$$

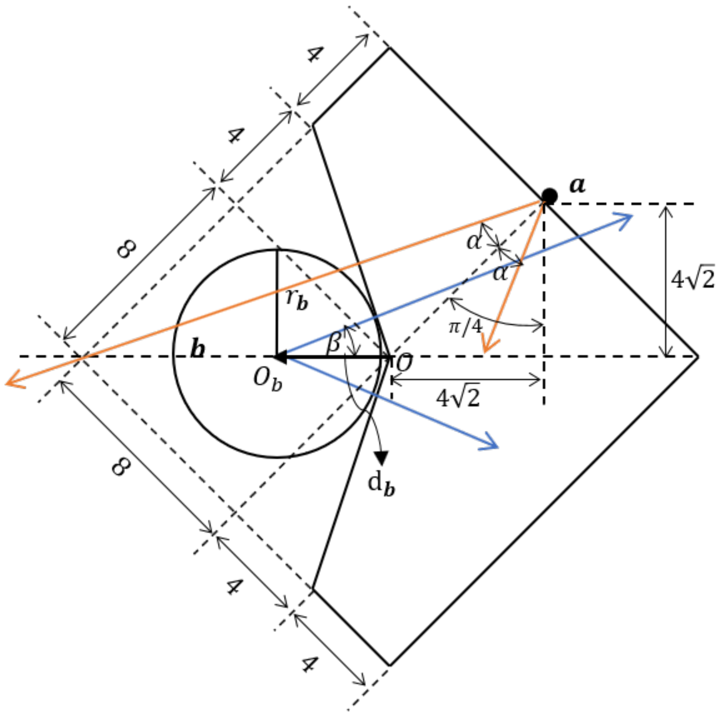


Figure 2: Figure for Problem 2(c).

The requirement that **a** lie inside the friction cone at **b** can then be expressed as

$$\frac{4\sqrt{2}}{\tan\beta} = 12\sqrt{2} < 4\sqrt{2} + d_{\mathbf{b}} \Rightarrow \frac{24}{\sqrt{5}} < r_{\mathbf{b}}.$$

The requirement that $O_{\mathbf{b}}$ lie inside the friction cone at **a** can be expressed as

$$\frac{4\sqrt{2}}{\tan(\frac{\pi}{4} - \alpha)} < 4\sqrt{2} + d_{\mathbf{b}} < \frac{4\sqrt{2}}{\tan(\frac{\pi}{4} + \alpha)} \Rightarrow 0 < r_{\mathbf{b}} < \frac{24}{\sqrt{5}}.$$

There is no solution $r_{\mathbf{b}}$ that simultaneously satisfies both inequalities.

Problem 3

(a) Using the general relations $a \times b = [a]b$ and $[Rw] = R[w]R^T$ for $a, b, \omega \in \mathbb{R}^3$ and $R \in SO(3)$,

$$\begin{aligned}
(q_1' - q_3') \times (q_2' - q_3') &= (Rq_1 + p - Rq_3 - p) \times (Rq_2 + p - Rq_3 - p) \\
&= R(q_1 - q_3) \times R(q_2 - q_3) \\
&= [R(q_1 - q_3)]R(q_2 - q_3) \\
&= R[q_1 - q_3]R^T R(q_2 - q_3) \\
&= R[q_1 - q_3](q_2 - q_3) \\
&= R((q_1 - q_3) \times (q_2 - q_3)).
\end{aligned}$$

(b) From the figure,

$$R_{01} = \mathrm{Rot}(\hat{z}, t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_{12} = I, \quad p_{01} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \quad p_{12} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}.$$

From the given T_{0p} we get

$$T_{p0} \;=\; T_{0p}^{-1} = \begin{bmatrix} 0 & 0 & 1 & -3 \\ -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}. ,$$

Using the relation $T_{p2} = T_{p0}T_{01}T_{12}$,

$$\begin{aligned} T_{p2} &= \begin{bmatrix} 0 & 0 & 1 & -3 \\ -1 & 0 & 0 & -2 \\ 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos t & -\sin t & 0 & 0 \\ \sin t & \cos t & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\cos t & \sin t & 0 & 2\sin t + 2 \\ -\sin t & -\cos t & 0 & 5 - 2\cos t \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

(c) p_{p0}, p_{p2} are known from (b), and p_{30}, p_{32}, p_{3p} are given. Then

$$\begin{aligned} R_{p3}(p_{30} - p_{3p}) &= p_{p0} \\ R_{p3}(p_{32} - p_{3p}) &= p_{p2}. \end{aligned}$$

A third equation can be obtained from (a):

$$R_{p3}((p_{30} - p_{3p}) \times (p_{32} - p_{3p})) = p_{p0} \times p_{p2}.$$

Combining the above three equations into the single matrix equation

$$R_{p3}\begin{bmatrix}(p_{30} - p_{3p}) & (p_{32} - p_{3p}) & (p_{30} - p_{3p}) \times (p_{32} - p_{3p})\end{bmatrix} = \begin{bmatrix}p_{p0} & p_{p2} & p_{p0} \times p_{p2}\end{bmatrix},$$

a unique R_{p3} can be obtained as follows:

$$R_{p3} = \begin{bmatrix}p_{p0} & p_{p2} & p_{p0} \times p_{p2}\end{bmatrix}\begin{bmatrix}(p_{30} - p_{3p}) & (p_{32} - p_{3p}) & (p_{30} - p_{3p}) \times (p_{32} - p_{3p})\end{bmatrix}^{-1}.$$

The above solution exists if and only if $(p_{30} - p_{3p})$ and $(p_{32} - p_{3p})$ are linearly independent.
(d) Since $r_{11}^2 + r_{12}^2 + r_{13}^2 = 1$ and $r_{13}^2 + r_{23}^2 + r_{33}^2 = 1$, R_{04} is of the form

$$R_{04} = \begin{bmatrix} 0 & 0 & 1 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & 0 \end{bmatrix}.$$

Since $R_{05} = I = R_{04}R_{45}$, it follows that

$$R_{45} = R_{04}^T = \begin{bmatrix} 0 & r_{21} & r_{31} \\ 0 & r_{22} & r_{32} \\ 1 & 0 & 0 \end{bmatrix}.$$

R_{45} can also be derived from the given $\hat{\omega}_3$ as

$$\begin{aligned} R_{45} &= \mathrm{Rot}(\hat{z}, \alpha)\mathrm{Rot}(\hat{y}, \beta)\mathrm{Rot}(\hat{\omega}_3, \gamma) \\ &= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} e^{[\hat{\omega}_3]\gamma}, \end{aligned}$$

where

$$e^{[\hat{\omega}_3]\gamma} = I + \sin\gamma[\hat{\omega}_3] + (1-\cos\gamma)[\hat{\omega}_3]^2 = \begin{bmatrix} \cos\gamma & -\frac{\sqrt{2}}{2}\sin\gamma & \frac{\sqrt{2}}{2}\sin\gamma \\ \frac{\sqrt{2}}{2}\sin\gamma & \frac{1+\cos\gamma}{2} & \frac{1-\cos\gamma}{2} \\ -\frac{\sqrt{2}}{2}\sin\gamma & \frac{1-\cos\gamma}{2} & \frac{1+\cos\gamma}{2} \end{bmatrix}.$$

Therefore R_{45} is of the form

$$R_{45} = \begin{bmatrix} - & - & - \\ - & - & - \\ -c_\gamma s_\beta - \frac{\sqrt{2}}{2}c_\beta s_\gamma & \frac{\sqrt{2}}{2}s_\beta s_\gamma - c_\beta\frac{(c_\gamma-1)}{2} & -\frac{\sqrt{2}}{2}s_\beta s_\gamma + c_\beta\frac{(c_\gamma+1)}{2} \end{bmatrix},$$

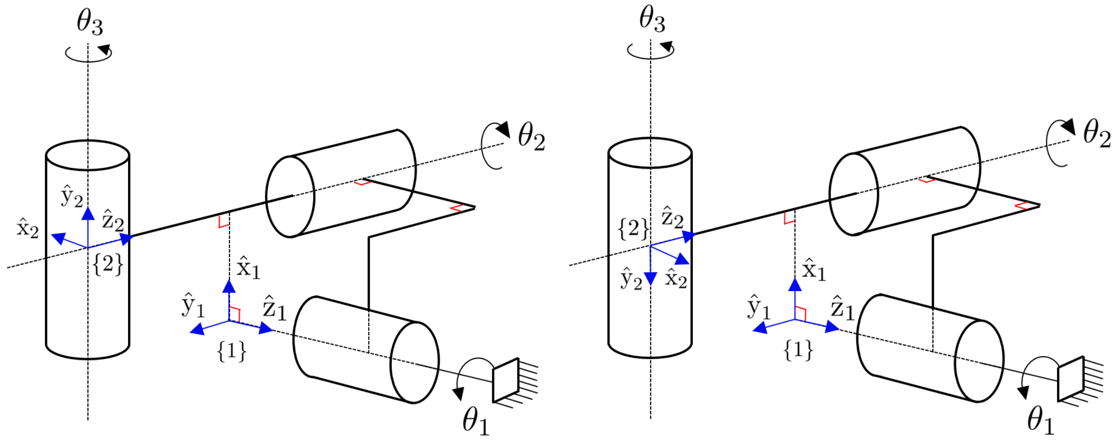
where c_γ denotes $\cos\gamma$, etc. Comparing this expression with the R_{45} obtained earlier, the following three equations in (β,γ) are obtained:

$$\begin{aligned} -c_\gamma s_\beta - \tfrac{\sqrt{2}}{2}c_\beta s_\gamma &= 1 \\ \tfrac{\sqrt{2}}{2}s_\beta s_\gamma - c_\beta\tfrac{(c_\gamma-1)}{2} &= 0 \\ -\tfrac{\sqrt{2}}{2}s_\beta s_\gamma + c_\beta\tfrac{(c_\gamma+1)}{2} &= 0 \end{aligned}.$$

Adding the second and third equations and solving for (β,γ) within the given interval leads to the unique solution $(\beta,\gamma) = (\frac{\pi}{2}, 0)$; since this solution fails to satisfy the first equation, we can conclude that no solution (α,β,γ) exists.

Problem 4

(a) Since the rotation axes for θ_2 and θ_3 intersect, two possible choices for frame $\{2\}$ exist:



(a) Case 1   (b) Case 2

Figure 3: Attached Denavit-Hartenberg frames.

The corresponding D-H parameters for the two cases are as follows:

(b) For this problem it is helpful to attach two auxiliary frames $\{4\}$ and $\{5\}$ before deriving T_{3w}. Set the origins of each frame to the circle centers, with $\hat{y}_4$ and $\hat{y}_5$ directed toward the origins of

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	$0°$	0	-L_3	$90° + \theta_1$
2	$90°$	L_1	$-(L_4 - L_2)$	$-90° + \theta_2$
3	$-90°$	0	0	$180° + \theta_3$

Table 1: D-H parameters corresponding to Case 1.

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	$0°$	0	-L_3	$90° + \theta_1$
2	$90°$	L_1	$-(L_4 - L_2)$	$90° + \theta_2$
3	$90°$	0	0	θ_3

Table 2: D-H parameters corresponding to Case 2.

frames $\{5\}$ and $\{w\}$, respectively. Then $T_{3w} = T_{34}T_{45}T_{5w}$, with

$$T_{34} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\cos\theta_4 & -\sin\theta_4 & 0 & 0 \\ \sin\theta_4 & -\cos\theta_4 & 0 & -L_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{45} = \begin{bmatrix} \cos\theta_4 & \sin\theta_4 & 0 & 0 \\ -\sin\theta_4 & \cos\theta_4 & 0 & 2R \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T_{5w} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$\therefore T_{3w} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\cos 2\theta_4 & -\sin 2\theta_4 & 0 & -2R\sin\theta_4 - L_6\sin 2\theta_4 \\ \sin 2\theta_4 & -\cos 2\theta_4 & 0 & -L_5 - 2R\cos\theta_4 - L_6\cos 2\theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

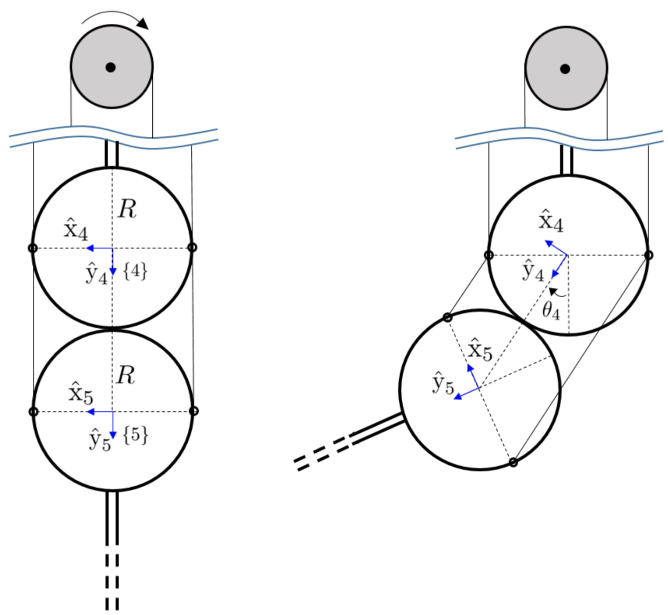
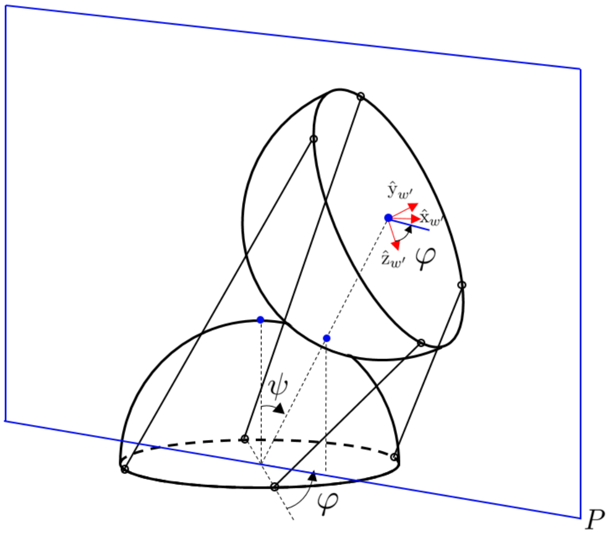
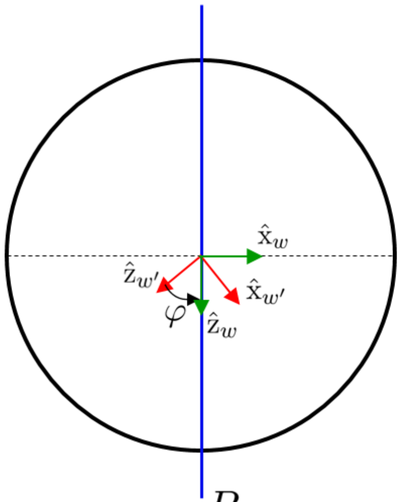


Figure 4: Two auxiliary frames for Problem 4(b).

(c) Denote by P the plane containing the center of the sphere (below), the current contact point, and the initial contact point. The center of the upper sphere also lies on P.



(a) Ambidex wrist mechanism.



(b) Transformation after Step 4.

Figure 5: Figure for Problem 4(c).

$T_{ww'}$ can be derived by sequentially applying the following five transformations (all transformations are made with respect to the moving (body) frame axes):

1. Rotate around $\hat{y}_w$ by φ to make $\hat{x}_w$ normal to P.
2. Rotate around $\hat{x}_w$ by ψ to align $\hat{y}_w$ with the line connecting the two sphere centers.
3. Translate in the direction of $\hat{y}_w$ by $2r$ so that the origins of $\{w\}$ and $\{w'\}$ overlap.
4. Rotate around $\hat{x}_w$ by ψ to align $\hat{y}_w$ with $\hat{y}_{w'}$.
5. Rotate around $\hat{y}_w$ by $-\varphi$ to align the remaining axes.

The resulting transformation Step 4 is shown in Figure 5(b). At this point we have $\hat{y}_w$ aligned with $\hat{y}_{w'}$, $\{\hat{x}_w, \hat{z}_w, \hat{x}_{w'}, \hat{z}_{w'}\}$ all lying on the same plane, and angle φ between $\hat{z}_w$ and $\hat{z}_{w'}$. Rotating $\{w\}$ around $\hat{y}_w$ by $-\varphi$ (Step 5) then will make $\{w\}$ and $\{w'\}$ overlap identically, resulting in

$$T_{ww'} = \mathrm{Rot}(\hat{y}_w, \varphi) \cdot \mathrm{Rot}(\hat{x}_w, \psi) \cdot \mathrm{Trans}(\hat{y}_w, 2r) \cdot \mathrm{Rot}(\hat{x}_w, \psi) \cdot \mathrm{Rot}(\hat{y}_w, -\varphi).$$

M2794.0027 Introduction to Robotics
Midterm Examination 2
May 23, 2019
CLOSED BOOK, CLOSED NOTES

Problem 1

Figure 1 shows two modules for a new modular robot design.

(a) For the RP module of Figure 1(a), draw appropriate link frames and derive the Denavit-Hartenberg parameters.

(b) Again for the RP module of Figure 1(a), this time express the forward kinematics in the form

$$T_{s_1 b_1} = M_1 e^{[\mathcal{A}_1]\theta_1} M_2 e^{[\mathcal{A}_2]\theta_2}.$$

for the frames $\{s_1\}$ and $\{b_1\}$ as drawn in the figure. You may use the relation between the PoE and D-H representations to solve the problem.

(c) For the RPR module of Figure 1(b), express its forward kinematics in the form

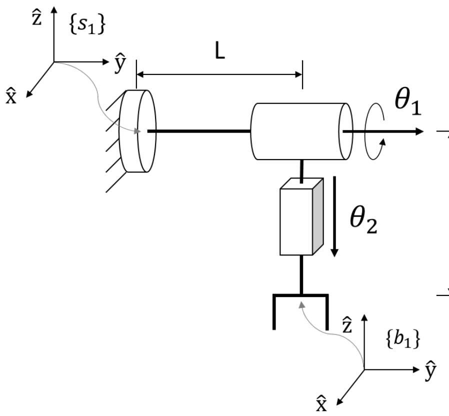
$$T_{s_2 b_2} = e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} M,$$

for frames $\{s_2\}$ and $\{b_2\}$ as drawn in the figure, where $M \in \mathrm{SE}(3)$ and $\mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5 \in \mathrm{se}(3)$.

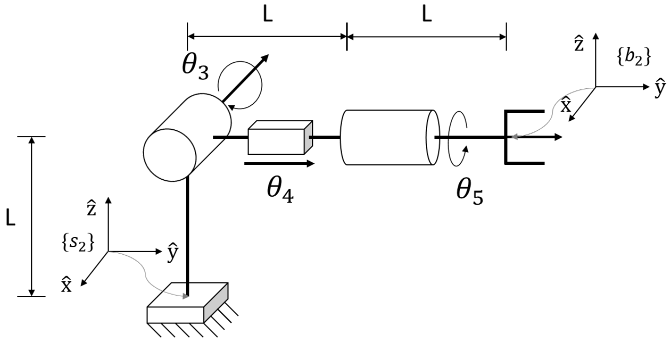
(d) The two modules are now connected as shown in Figure 1(c), so that the last link of Figure 1(a) and the first link of Figure 1(b) are collinear. The forward kinematics of the entire RPRPR robot can then be expressed in the form

$$T_{s_1 b_2} = e^{[\mathcal{S}'_1]\theta_1} e^{[\mathcal{S}'_2]\theta_2} \cdots e^{[\mathcal{S}'_5]\theta_5} M'.$$

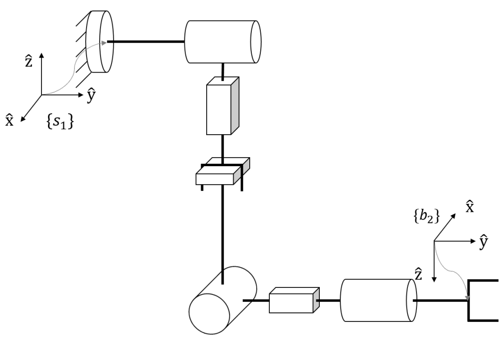
First find $T_{b_1 s_2}$, then express M' and $\mathcal{S}'_1, \ldots, \mathcal{S}'_5$ in terms of $T_{b_1 s_2}$,M, M_1, M_2, $\mathcal{A}_1, \mathcal{A}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$.



(a) RP module



(b) RPR module



(c) The two modules connected

Figure 1: Figure for Problem 1

Problem 2
(a) Let $J_s(\theta)$ be the space Jacobian for an n-link open chain, and $J_b(\theta)$ be its body Jacobian. Show that for any $T \in SE(3)$, the matrix $[\mathrm{Ad}_T]$ is nonsingular. Use this result to show that $J_s(\theta)$ and $J_b(\theta)$ always have the same rank regardless of θ.
(b) Now consider the RRPRRR arm of Figure 2. Find at least three singularities of this arm, and describe the screw conditions for each singularity.
(c) Let frame $\{3\}$ be attached to the third link, and V_3 be the spatial velocity of the end-effector frame expressed in frame $\{3\}$ (that is, $V_3 = J_3(\theta)\dot{\theta}$). Derive $J_3(\theta)$.

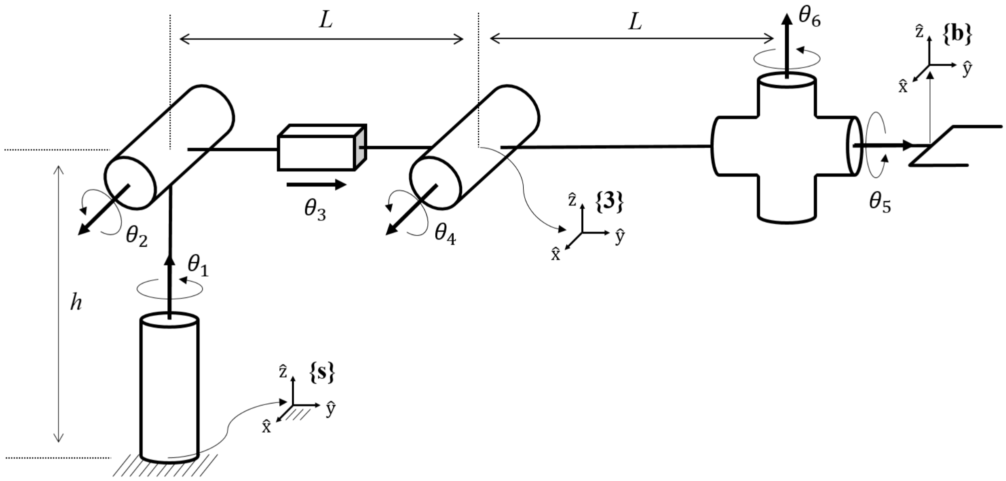


Figure 2: RRPRRR arm for Problem 2 shown in its zero position

Problem 3

(a) The RRRRR robot arm of Figure 3 is shown its zero position. For the end-effector configuration

$$T = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & \cos\alpha & -\sin\alpha & p_y \\ 0 & \sin\alpha & \cos\alpha & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where α and (p_x, p_y, p_z) are given, how many inverse kinematic solutions will there be in general? Explain your answer using geometric reasoning; you do not need to solve the inverse kinematics explicitly for this part.

(b) For the same T given in (a), find inverse kinematic solutions for θ_4 and θ_5 only.

(c) Let $\theta_1 = \theta_2 = \theta_3 = 0$ and $p_x = 0$. Given arbitrary (p_y, p_z), write down the Newton-Raphson iteration for numerically solving the inverse kinematics for (θ_4, θ_5).

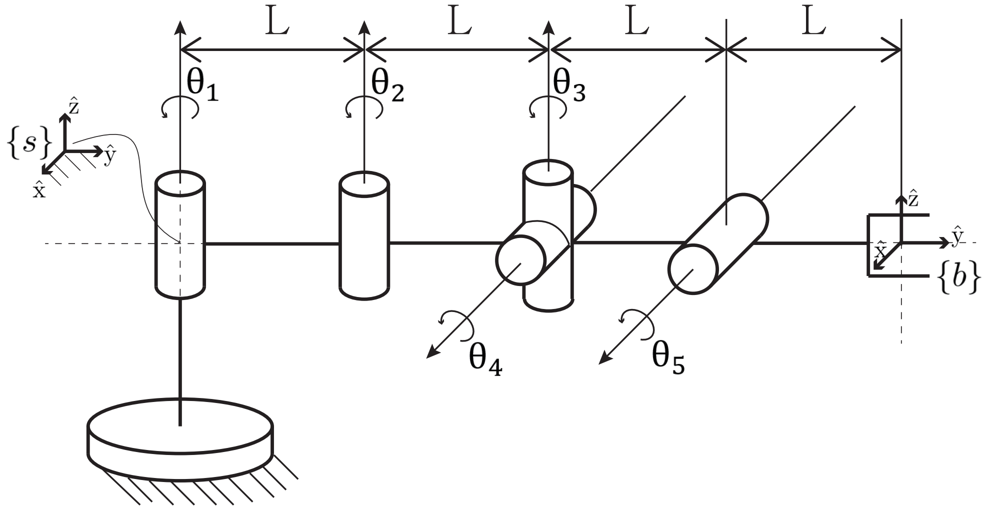


Figure 3: RRRRR arm for Problem 3 shown in its zero position

Problem 4

The RRRR arm of Figure 4 is shown in its zero position.

(a) At the configuration $\theta_2 = \pi/2$, $\theta_1 = \theta_3 = \theta_4 = 0$, an external spatial force (or wrench) $\mathcal{F}_b = (1, -1, 0, -1, -1, -1)^\top$ is applied to the end-effector. Find the joint torque τ to keep the arm in static equilibrium.

(b) At the same joint configuration $\theta_2 = \pi/2$, $\theta_1 = \theta_3 = \theta_4 = 0$, this time assume the external wrench is of the form $\mathcal{F}_b = (0, 0, 0, f_x, f_y, f_z)^\top$. Given the joint torque constraint $\tau^\top \tau = 1$, we wish to find the optimal τ that maximizes ${f_x}^2 + {f_y}^2 + {f_z}^2$. Derive the first-order necessary conditions for this optimization problem in terms of (f_x, f_y, f_z) and the Lagrange multiplier λ. (You do not need to solve these equations for this part.)

(c) For the first-order necessary conditions you derived in (b), now perform one iteration of the Newton-Raphson method for the initial guess $(f_x, f_y, f_z, \lambda) = (1, 3, 1, 2)$. You do not need to explicitly evaluate matrix inverses. If you were unable to derive the first-order necessary conditions, assume they are of the general form $h_i(f_x, f_y, f_z, \lambda) = 0$, $i = 1, 2, \ldots$, and derive the procedure.

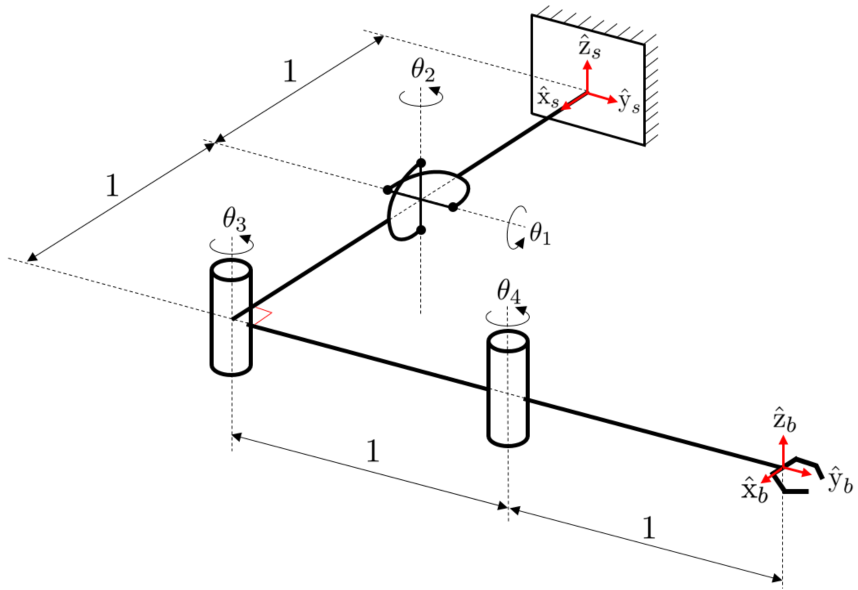


Figure 4: RRRR arm for Problem 4 shown in its zero position

M2794.0027 Introduction to Robotics
Midterm Examination 2 Solutions
May 23, 2019

Problem 1

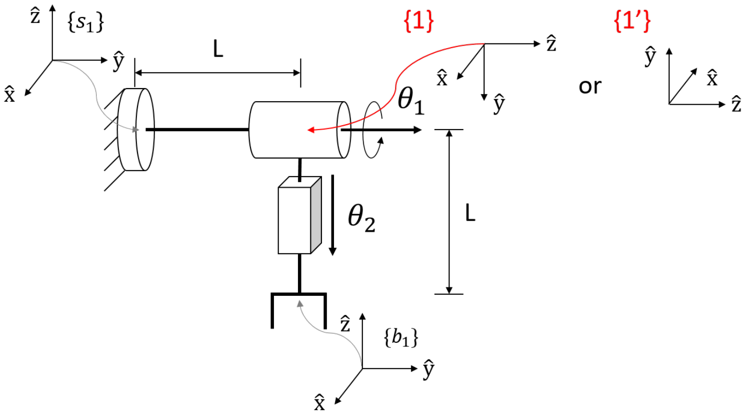


Figure 1: RP module of Problem 1

(a) Figure 1 shows two possible choices of link frame for link 1. The D-H parameters using frame {1} are

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	$-\frac{\pi}{2}$	0	L	θ_1
b_1	$\frac{\pi}{2}$	0	$-L-\theta_2$	0

The D-H parameters using frame {1′} are,

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	$-\frac{\pi}{2}$	0	L	$\theta_1+\pi$
b_1	$-\frac{\pi}{2}$	0	$-L-\theta_2$	π

(b) Using the Denavit-Hartenberg parameters derived in (a), we can compute $M_1, M_2, \mathcal{A}_1, \mathcal{A}_2$ as follows:

When frame $\{1\}$ is used,

$$
\begin{aligned}
M_1 &= \text{Rot}\left(\hat{\text{x}}, -\frac{\pi}{2}\right) \cdot \text{Trans}\left(\hat{\text{x}}, 0\right) \cdot \text{Trans}\left(\hat{\text{z}}, L\right) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
M_2 &= \text{Rot}\left(\hat{\text{x}}, \frac{\pi}{2}\right) \cdot \text{Trans}\left(\hat{\text{x}}, 0\right) \cdot \text{Trans}\left(\hat{\text{z}}, -L\right) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\mathcal{A}_1 &= \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^T \\
\mathcal{A}_2 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}^T.
\end{aligned}
$$

When frame $\{1'\}$ is used,

$$
\begin{aligned}
M_1 &= \text{Rot}\left(\hat{\text{x}}, -\frac{\pi}{2}\right) \cdot \text{Trans}\left(\hat{\text{x}}, 0\right) \cdot \text{Trans}\left(\hat{\text{z}}, L\right) \cdot \text{Rot}\left(\hat{\text{z}}, \pi\right) \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
M_2 &= \text{Rot}\left(\hat{\text{x}}, -\frac{\pi}{2}\right) \cdot \text{Trans}\left(\hat{\text{x}}, 0\right) \cdot \text{Trans}\left(\hat{\text{z}}, -L\right) \cdot \text{Rot}\left(\hat{\text{z}}, \pi\right) \\
&= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

The screws $\mathcal{A}_1$ and $\mathcal{A}_2$ are the same for both choices of link frame.
(c) The forward kinematics for the RPR module can be expressed in product-of-exponentials form using the following values for $\hat{\omega}_i, v_i$ and M:

i	$\hat{\omega}_i$	q_i	v_i
3	$(-1,0,0)$	$(0,0,L)$	$(0,-L,0)$
4	$(0,0,0)$	-	$(0,1,0)$
5	$(0,1,0)$	$(0,0,L)$	$(-L,0,0)$

$$
\begin{aligned}
\mathcal{S}_3 &= \begin{bmatrix} -1 & 0 & 0 & 0 & -L & 0 \end{bmatrix}^T \\
\mathcal{S}_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}^T \\
\mathcal{S}_5 &= \begin{bmatrix} 0 & 1 & 0 & -L & 0 & 0 \end{bmatrix}^T \\
M &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2L \\ 0 & 0 & 1 & L \\ 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}
$$

(d) Since frames $\{s_2\}$ and $\{b_1\}$ have overlapping origins, it follows that $p_{b_1 s_2} = 0$. To ensure that the last link of the RP module and the first link of the RPR module are collinear, frame $\{s_2\}$ should be rotated by angle π around the ŷ-axis of frame $\{b_1\}$, resulting in

$$T_{b_1 s_2} = \begin{bmatrix} \mathrm{Rot}(\hat{\mathrm{y}}, \pi) & 0 \\ 0 & 1 \end{bmatrix}.$$

$T_{s_1 b_2}$ can then be expressed in product-of-exponentials form using $T_{s_1 b_1}$ and $T_{s_2 b_2}$:

$$\begin{aligned}
T_{s_1 b_2} &= T_{s_1 b_1} T_{b_1 s_2} T_{s_2 b_2} \\
&= M_1 e^{[\mathcal{A}_1]\theta_1} M_2 e^{[\mathcal{A}_2]\theta_2} T_{b_1 s_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} M \\
&= \left(M_1 e^{[\mathcal{A}_1]\theta_1} M_1^{-1}\right) \left(M_1 M_2 e^{[\mathcal{A}_2]\theta_2} M_2^{-1} M_1^{-1}\right) \left(M_1 M_2 T_{b_1 s_2} e^{[\mathcal{S}_3]\theta_3} T_{b_1 s_2}^{-1} M_2^{-1} M_1^{-1}\right) \\
&\quad \cdots \left(M_1 M_2 T_{b_1 s_2} e^{[\mathcal{S}_5]\theta_5} T_{b_1 s_2}^{-1} M_2^{-1} M_1^{-1}\right) \left(M_1 M_2 T_{b_1 s_2} M\right) \\
&= e^{M_1 [\mathcal{A}_1] M_1^{-1} \theta_1} e^{(M_1 M_2)[\mathcal{A}_2](M_1 M_2)^{-1}\theta_2} e^{\left(M_1 M_2 T_{b_1 s_2}\right)[\mathcal{S}_3]\left(M_1 M_2 T_{b_1 s_2}\right)^{-1}\theta_3} \\
&\quad \cdots e^{\left(M_1 M_2 T_{b_1 s_2}\right)[\mathcal{S}_5]\left(M_1 M_2 T_{b_1 s_2}\right)^{-1}\theta_5} \left(M_1 M_2 T_{b_1 s_2} M\right) \\
&= e^{[\mathcal{S}'_1]\theta_1} e^{[\mathcal{S}'_2]\theta_2} \cdots e^{[\mathcal{S}'_5]\theta_5} M.
\end{aligned}$$

From the above we have

$$\begin{aligned}
\mathcal{S}'_1 &= \mathrm{Ad}_{M_1}(\mathcal{A}_1) \\
\mathcal{S}'_2 &= \mathrm{Ad}_{M_1 M_2}(\mathcal{A}_2) \\
\mathcal{S}'_3 &= \mathrm{Ad}_{M_1 M_2 T_{b_1 s_2}}(\mathcal{S}_3) \\
&\vdots \\
\mathcal{S}'_5 &= \mathrm{Ad}_{M_1 M_2 T_{b_1 s_2}}(\mathcal{S}_5) \\
M' &= M_1 M_2 T_{b_1 s_2} M.
\end{aligned}$$

Problem 2

(a) There are several ways to show that $[\mathrm{Ad}_T]$ is nonsingular, e.g., calculating its determinant and showing that it is always nonzero, showing that the linear equation $[\mathrm{Ad}_T]V = 0$ has the unique solution $V = (0, 0)$, or directly calculating the inverse $[\mathrm{Ad}_T]^{-1}$ and showing that it always exists; here we'll adopt the latter approach. Letting

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix},$$

we have

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix}, \quad [\mathrm{Ad}_{T^{-1}}] = \begin{bmatrix} R^T & 0 \\ [-R^T p]\,R^T & R^T \end{bmatrix} = \begin{bmatrix} R^T & 0 \\ -R^T\,[p] & R^T \end{bmatrix}.$$

Multiplying $[\mathrm{Ad}_T]$ and $[\mathrm{Ad}_{T^{-1}}]$,

$$\begin{aligned} [\mathrm{Ad}_T][\mathrm{Ad}_{T^{-1}}] &= \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix} \begin{bmatrix} R^T & 0 \\ -R^T\,[p] & R^T \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ [p] + [-p] & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \end{aligned}$$

from which it follows that $[\mathrm{Ad}_T]^{-1} = [\mathrm{Ad}_{T^{-1}}]$ always exists. The relationship between the space Jacobian J_s and body Jacobian J_b can then be expressed as follows (here let $T_{sb} = T$):

$$J_s(\theta) = [\mathrm{Ad}_T] J_b(\theta).$$

Since $[\mathrm{Ad}_T]$ is nonsingular, from a standard linear algebraic property of the rank of matrix products,

$$\mathrm{rank}(J_s) = \mathrm{rank}([\mathrm{Ad}_T] J_b) = \mathrm{rank}(J_b),$$

so that $J_s(\theta)$ and $J_b(\theta)$ always have the same rank for all θ.

(b) The arm depicted has at least four readily identifiable singularities (and possibly more):

- Two collinear revolute joints:

$$\begin{aligned} \theta_3 = -L \quad &: \text{ joints 2, 4 are collinear.} \\ \theta_3 = 0,\ \theta_4 = \pi,\ \theta_5 = \tfrac{\pi}{2} \quad &: \text{ joints 2, 6 are collinear.} \end{aligned}$$

- Three parallel and coplanar revolute joints:

$$\theta_4 = 0,\ \theta_5 = \tfrac{\pi}{2}, \tfrac{3\pi}{2} \quad : \text{ joints 2, 4, 6 are parallel and coplanar.}$$

- Four revolute joints intersecting at a common point:

$$\theta_3 = 0,\ \theta_4 = \pi \quad : \text{ joints 1, 2, 5, 6 intersect at a common point.}$$

- Four coplanar revolute joints:

$$\theta_4 = 0,\ \theta_5 = \tfrac{\pi}{2}, \tfrac{3\pi}{2} \quad : \text{ joints 2, 4, 5, 6 are coplanar.}$$

(c) First derive T_{3b} in product of exponentials form: Letting C_i be the screw vector for joint i, $i = 1, \ldots, 6$, we have

$$T_{3b} = e^{[C_4]\theta_4} e^{[C_5]\theta_5} e^{[C_6]\theta_6} M_{3b}$$

and $J_3 = [\mathrm{Ad}_{T_{3b}}]J_b$. The first column of J_3, denoted C_1', is given by

$$\begin{aligned} C_1' &= [\mathrm{Ad}_{T_{3b}}]B_1' \\ &= [\mathrm{Ad}_{e^{[C_4]\theta_4} e^{[C_5]\theta_5} e^{[C_6]\theta_6} M_{3b} e^{-[B_6]\theta_6} e^{-[B_5]\theta_5} e^{-[B_4]\theta_4} e^{-[B_3]\theta_3} e^{-[B_2]\theta_2}}]B_1 \\ &= [\mathrm{Ad}_{e^{[C_4]\theta_4} e^{[C_5]\theta_5} e^{[C_6]\theta_6} e^{-[C_6]\theta_6} e^{-[C_5]\theta_5} e^{-[C_4]\theta_4} e^{-[C_3]\theta_3} e^{-[C_2]\theta_2}}]C_1 \\ &= [\mathrm{Ad}_{e^{-[C_3]\theta_3} e^{-[C_2]\theta_2}}]C_1, \end{aligned}$$

where B_1' denotes the first column of J_b. The other columns of J_3 can be derived in a similar fashion:

$$J_3 = \left[\; [\mathrm{Ad}_{e^{-[C_3]\theta_3} e^{-[C_2]\theta_2}}]C_1 \quad [\mathrm{Ad}_{e^{-[C_3]\theta_3}}]C_2 \quad C_3 \quad C_4 \quad [\mathrm{Ad}_{e^{[C_4]\theta_4}}]C_5 \quad [\mathrm{Ad}_{e^{[C_4]\theta_4} e^{[C_5]\theta_5}}]C_6 \; \right].$$

J_3 can then be expressed in terms of $(\theta_1, \ldots, \theta_6)$ as follows (c_i, s_i denote $\cos\theta_i, \sin\theta_i$):

- C_1':

$$\hat{\omega}_1' = \mathrm{Rot}(\hat{\mathrm{x}}, -\theta_2) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ s_2 \\ c_2 \end{bmatrix}, \; v_1' = -\hat{\omega}_1' \times \begin{bmatrix} 0 \\ -L - \theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -(L+\theta_3)c_2 \\ 0 \\ 0 \end{bmatrix}$$

- C_2':

$$\hat{\omega}_2' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; v_2' = -\hat{\omega}_2' \times \begin{bmatrix} 0 \\ -L - \theta_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ L+\theta_3 \end{bmatrix}$$

- C_3', C_4':

$$\hat{\omega}_3' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; v_3' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{\omega}_4' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; v_4' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- C_5':

$$\hat{\omega}_5' = \mathrm{Rot}(\hat{\mathrm{x}}, \theta_4) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ c_4 \\ s_4 \end{bmatrix}, \; v_5' = -\hat{\omega}_5' \times \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- C_6':

$$\hat{\omega}_6' = \mathrm{Rot}(\hat{\mathrm{x}}, \theta_4)\mathrm{Rot}(\hat{\mathrm{y}}, \theta_5) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} s_5 \\ -s_4 c_5 \\ c_4 c_5 \end{bmatrix}, \; q_6' = \mathrm{Rot}(\hat{\mathrm{x}}, \theta_4) \begin{bmatrix} 0 \\ L \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ Lc_4 \\ Ls_4 \end{bmatrix}$$

$$v_6' = -\hat{\omega}_6' \times \begin{bmatrix} 0 \\ Lc_4 \\ Ls_4 \end{bmatrix} = \begin{bmatrix} Lc_5 \\ Ls_4 s_5 \\ -Lc_4 s_5 \end{bmatrix}$$

$$\therefore J_3 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & s_5 \\ s_2 & 0 & 0 & 0 & c_4 & -s_4 c_5 \\ c_2 & 0 & 0 & 0 & s_4 & c_4 c_5 \\ -(L+\theta_3)c_2 & 0 & 0 & 0 & 0 & Lc_5 \\ 0 & 0 & 1 & 0 & 0 & Ls_4 s_5 \\ 0 & L+\theta_3 & 0 & 0 & 0 & -Lc_4 s_5 \end{bmatrix}.$$

Problem 3

(a) The desired end-effector position and orientation is given by

$$T = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & \cos\alpha & -\sin\alpha & p_y \\ 0 & \sin\alpha & \cos\alpha & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since the end-effector orientation is of the form $\mathrm{Rot}(\hat{\mathrm{x}}, \alpha)$, the following two conditions must always hold:

$$\begin{aligned} \theta_1 + \theta_2 + \theta_3 &= 0 \\ \theta_4 + \theta_5 &= \alpha. \end{aligned}$$

In Figures 2 and 3 below, let A, B, C, D denote the locations of joints 1, 2, 3, 5, and let E denote the location of the end-effector center. Given p_z, there exist two possible solution pairs for (θ_4, θ_5) as illustrated in Figure 2 by (C_1, D) and (C_2, D). Given the two possible locations C_1 and C_2 for joint 3, there exist elbow-up and elbow-down solutions for each C_i, resulting in a total of four inverse kinematic solutions as illustrated in Figure 3.

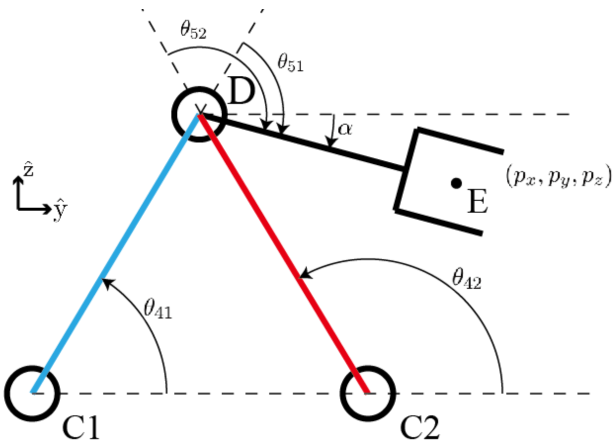


Figure 2: Robot arm as seen from the $\hat{\mathrm{x}}$ direction

(b) θ_4 and θ_5 are determined from p_z and α:

$$\begin{aligned} \theta_4 + \theta_5 &= \alpha \\ \sin\theta_4 + \sin(\theta_4 + \theta_5) &= p_z. \end{aligned}$$

Solving the two equations for θ_4 and θ_5,

$$\begin{aligned} \theta_4 &= \arcsin\left(\frac{p_z}{L} - \sin\alpha\right) \\ \theta_5 &= \alpha - \arcsin\left(\frac{p_z}{L} - \sin\alpha\right) \end{aligned}$$

or

$$\begin{aligned} \theta_4 &= \pi - \arcsin\left(\frac{p_z}{L} - \sin\alpha\right) \\ \theta_5 &= \alpha - \pi + \arcsin\left(\frac{p_z}{L} - \sin\alpha\right). \end{aligned}$$

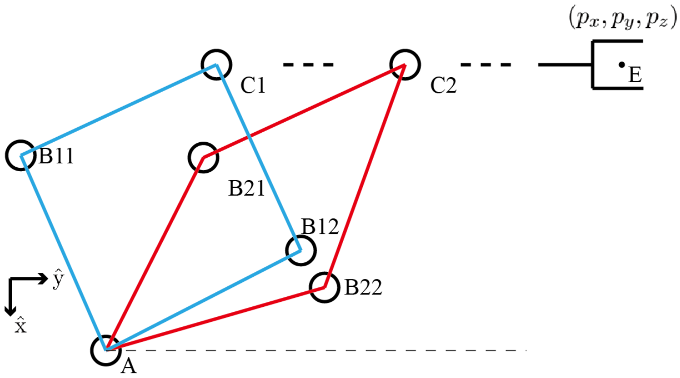


Figure 3: Robot arm as seen from the ẑ direction

(c) Let $\theta = (\theta_4, \theta_5)^T$ and define $p'(\theta)$ as

$$p'(\theta) = \begin{bmatrix} L\cos\theta_4 + L\cos(\theta_4+\theta_5) + 2L \\ L\sin\theta_4 + L\sin(\theta_4+\theta_5) \end{bmatrix}.$$

With given $p = (p_y, p_z)^T$, define

$$f(\theta) = p'(\theta) - p = \begin{bmatrix} L\cos\theta_4 + L\cos(\theta_4+\theta_5) + 2L - p_y \\ L\sin\theta_4 + L\sin(\theta_4+\theta_5) - p_z \end{bmatrix}.$$

The partial derivative of f at $\theta = \theta_k$ is given by

$$\frac{\partial f}{\partial \theta}(\theta_k) = L \begin{bmatrix} -\sin\theta_{4k} - \sin(\theta_{4k}+\theta_{5k}) & -\sin(\theta_{4k}+\theta_{5k}) \\ \cos\theta_{4k} + \cos(\theta_{4k}+\theta_{5k}) & \cos(\theta_{4k}+\theta_{5k}) \end{bmatrix}.$$

Its inverse is

$$\left(\frac{\partial f}{\partial \theta}(\theta_k)\right)^{-1} = \frac{1}{L\sin\theta_{5k}} \begin{bmatrix} cos(\theta_{4k}+\theta_{5k}) & \sin(\theta_{4k}+\theta_{5k}) \\ -\cos\theta_{4k} - \cos(\theta_{4k}+\theta_{5k}) & -\sin\theta_{4k} - \sin(\theta_{4k}+\theta_{5k}) \end{bmatrix},$$

so that the Newton-Raphson iteration for the inverse kinematics is

$$(\theta_{4(k+1)}, \theta_{5(k+1)})^T = (\theta_{4k}, \theta_{5k})^T - \left(\frac{\partial f}{\partial \theta}(\theta_k)\right)^{-1} f(\theta_k).$$

Problem 4

(a) To keep the arm in static equilibrium, the wrench $-\mathcal{F}_b = [-1, 1, 0, 1, 1, 1]^\top$ needs to be generated. The body Jacobian in the given configuration $\theta_2 = \frac{\pi}{2}, \theta_1 = \theta_3 = \theta_4 = 0$ is

$$J_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}.$$

The required joint torque τ is therefore

$$\tau = J_b^\top(-\mathcal{F}_b) = [1, -1, -2, -1]^\top.$$

(b) With $\mathcal{F}_b = [0, 0, 0, f_x, f_y, f_z]^\top$, the static torque-wrench equation can be reduced as follows:

$$\tau = -\begin{bmatrix} 0 & -2 & -2 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}^\top \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}.$$

Letting $f \triangleq [f_x, f_y, f_z]^\top$, we have

$$\tau^\top\tau = 1 \quad \Longrightarrow \quad f^\top \begin{bmatrix} 0 & -2 & -2 & -1 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ -2 & 1 & 0 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} f = f^\top \begin{bmatrix} 9 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} f.$$

Letting $Q \triangleq \begin{bmatrix} 9 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, the given optimization problem can be stated as

$$\min_f -f^\top f \quad \text{subject to } f^\top Q f = 1.$$

Define the Lagrangian $H(f, \lambda) = -f^\top f + \lambda(f^\top Q f - 1)$, $\lambda \in \mathbb{R}$. The first-order necessary conditions are then

$$\begin{aligned} \frac{\partial H}{\partial f} &= -2f^\top + 2\lambda f^\top Q = 0 \Longrightarrow (\lambda Q - I)f = 0, \\ \frac{\partial H}{\partial \lambda} &= f^\top Q f - 1 = 0. \end{aligned}$$

(c) Define

$$h(f, \lambda) \triangleq \begin{bmatrix} \frac{\partial H}{\partial f}(f, \lambda) \\ \frac{\partial H}{\partial \lambda}(f, \lambda) \end{bmatrix} = \begin{bmatrix} (9\lambda - 1)f_x - 2\lambda f_y \\ -2\lambda f_x + (\lambda - 1)f_y \\ (4\lambda - 1)f_z \\ 9{f_x}^2 - 4f_x f_y + {f_y}^2 + 4{f_z}^2 - 1 \end{bmatrix}.$$

Letting $\mathrm{x} = [f_x, f_y, f_z, \lambda]^\top$, one iteration of the Newton-Raphson method

$$\mathrm{x}_{n+1} = \mathrm{x}_n - \frac{\partial h}{\partial \mathrm{x}}^{-1}(\mathrm{x}_n) h(\mathrm{x}_n)$$

with initial condition $x_0 = [1, 3, 1, 2]$ results in

$$x_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 17 & -4 & 0 & 3 \\ -4 & 1 & 0 & 1 \\ 0 & 0 & 7 & 4 \\ 6 & 2 & 8 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ -1 \\ 7 \\ 9 \end{bmatrix}.$$

M2794.0027 Introduction to Robotics
Final Examination
June 11, 2019
CLOSED BOOK, CLOSED NOTES

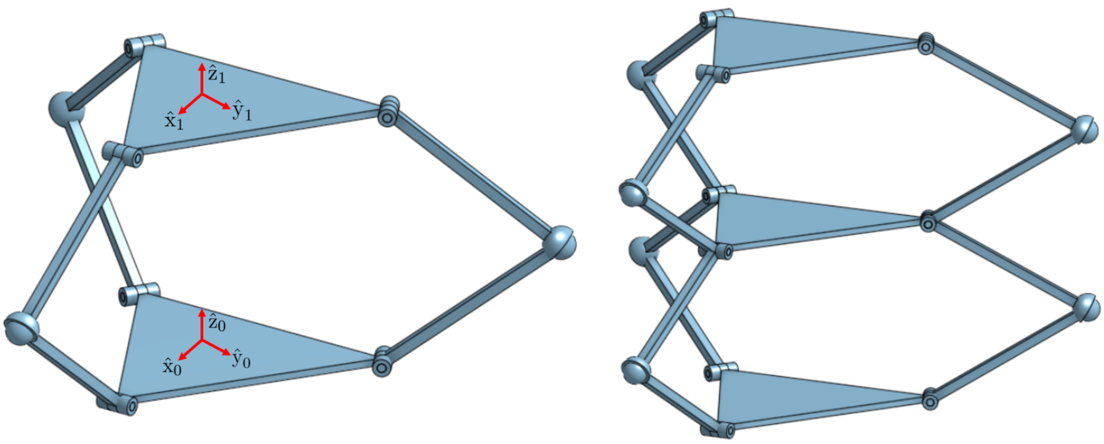
Problem 1 (45 points)
(a) The 3-RSR parallel mechanism of Figure 1(a) consists of two identical equilateral triangles connected by three RSR open chains, with all links of equal length. Frames {0} and {1} are attached to the respective centers of the two triangles. Assuming the lower triangle is fixed to ground, use Grübler's formula to calculate the degrees of freedom of the mechanism.
(b) Now suppose two 3-RSR parallel mechanisms are stacked on top of each other as shown in Figure 1(b), so that the three triangles are parallel and overlap each other exactly in the x-y plane; only the z-directional heights of the three triangles differ. Assume the top and lower triangles are fixed to ground, while the middle triangle can move. Use Grübler's formula to calculate the degrees of freedom of this mechanism. Does your answer agree with your intuition about this mechanism? Carefully explain your reasoning.
(c) Returning to the 3-RSR mechanism of Figure 1(a), a kinematic analysis shows that the set of all possible orientations R_{01} for this mechanism can be parametrized as follows using only two variables φ and ψ in the range $[-\pi, \pi]$:

$$R_{01}(\varphi, \psi) = \mathrm{Rot}(\hat{z}_1, \varphi)\mathrm{Rot}(\hat{y}_1, 2\psi)\mathrm{Rot}(\hat{z}_1, -\varphi).$$

Determine, if they exist, angles (φ, ψ) for the rotation

$$R_{01} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{bmatrix} = e^{[\hat{\omega}]\theta}, \quad \hat{\omega} = \frac{1}{\sqrt{3}}(1,1,1)^\top, \quad \theta = \frac{\pi}{3}.$$

If a solution (φ, ψ) does not exist for the R_{01} given above, carefully explain why.



(a) A 3-RSR parallel mechanism
(b) Two stacked 3-RSR mechanisms

Figure 1: Figures for Problem 1.

Problem 2 (45 points)
A rigid equilateral triangle with each side of length 2 is grasped by four frictionless point contacts $A(x_1)$, $B(x_2)$, $C(x_3)$, $D(x_4)$ as shown in Figure 2. The contacts are constrained to always be in contact with the corresponding edges, i.e., $0 < x_i < 2$, $i = 1, \ldots, 4$.

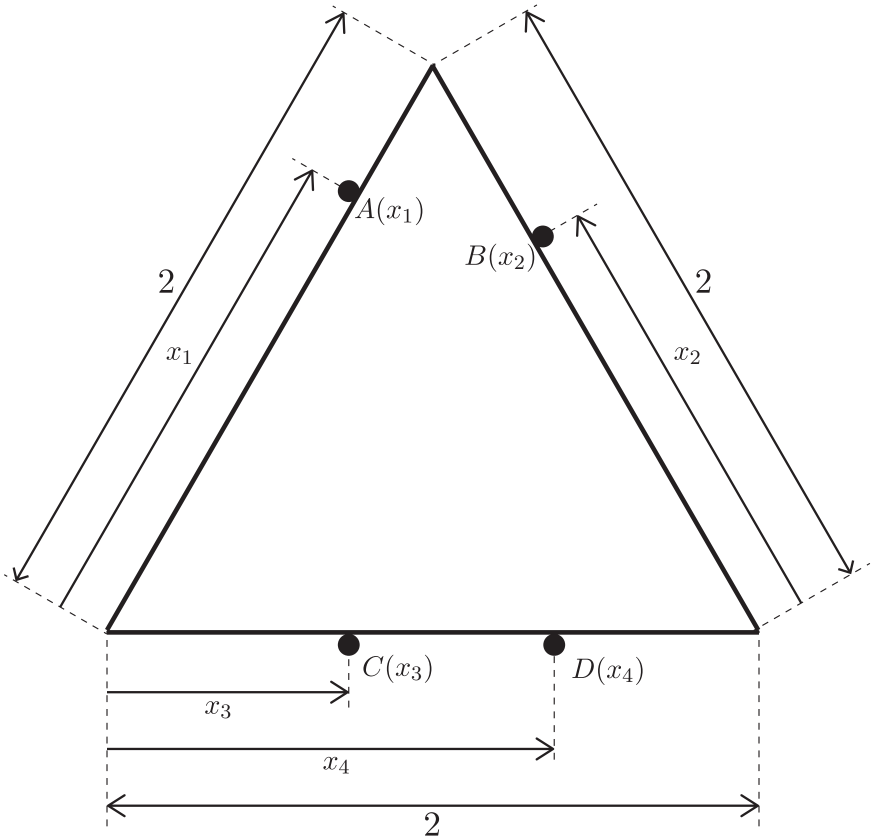


Figure 2: A rigid equilateral triangle constrained by point contacts.

(a) Let $x_1 = x_4 = \frac{3}{2}$ and $x_2 = x_3 = 1$. Determine if this grasp is a force closure grasp. Place your reference frame at contact C for any calculations that verify your answer.
(b) This time let x_1, x_2, x_3, x_4 be arbitrary values in the range $(0, 2)$. Show that the grasp is force closure if and only if the point of intersection between the lines of action at A and B lies between the lines of action at C and D. (*Hint*: Think carefully about where to place your reference frame; depending on your choice, the analysis can be made very simple).
(c) Now suppose contact $D(x_4)$ has been removed, and that only the three contacts $A(x_1)$, $B(x_2)$, $C(x_3)$ are in contact with the triangle. Let $x_1 = x_2 = x_3 = 1$, and suppose each point contact has identical frictional coefficient μ. For what range of values of μ is the triangle in force closure? You must explain your answer carefully to receive full credit.

Problem 3 (45 points)
Figure 3 shows an RRPRPR arm in its zero position, with fixed frame {s} attached to ground and link frames {b} and {c} attached to the robot as shown.
(a) Using {s} as the fixed frame but assigning appropriate link frames as needed, find the Denavit-Hartenberg parameters $(\alpha_{i-1}, a_{i-1}, d_i, \phi_i)$ for $i = 1, 2, 3$, subject to the requirement that $\alpha_{i-1} \in [0, \pi]$, $i = 1, 2, 3$.
(b) The forward kinematics T_{sb} can written in the following product of exponentials form:

$$T_{sb} = e^{[\mathcal{A}_1]\theta_1} e^{[\mathcal{A}_2]\theta_2} M_{sc} e^{[\mathcal{A}_3]\theta_3} e^{[\mathcal{A}_4]\theta_4} M_{cb} e^{[\mathcal{A}_5]\theta_5} e^{[\mathcal{A}_6]\theta_6}.$$

Find M_{sc}, M_{cb}, $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$.
(c) Referring to Figure 3, while the robot is in its zero position, an external force $f_c = (1, 1, 1)^T$ is applied to the origin of frame {c}, where f_c is expressed in frame {c} coordinates. At the same time, an external force $f_b = (1, -1, -1)^T$ is applied to the origin of frame {b}, where f_b is expressed in frame {b} coordinates. Find the joint torque $\tau = (\tau_1, \ldots, \tau_6)^T$ needed to keep the robot in static equilibrium.

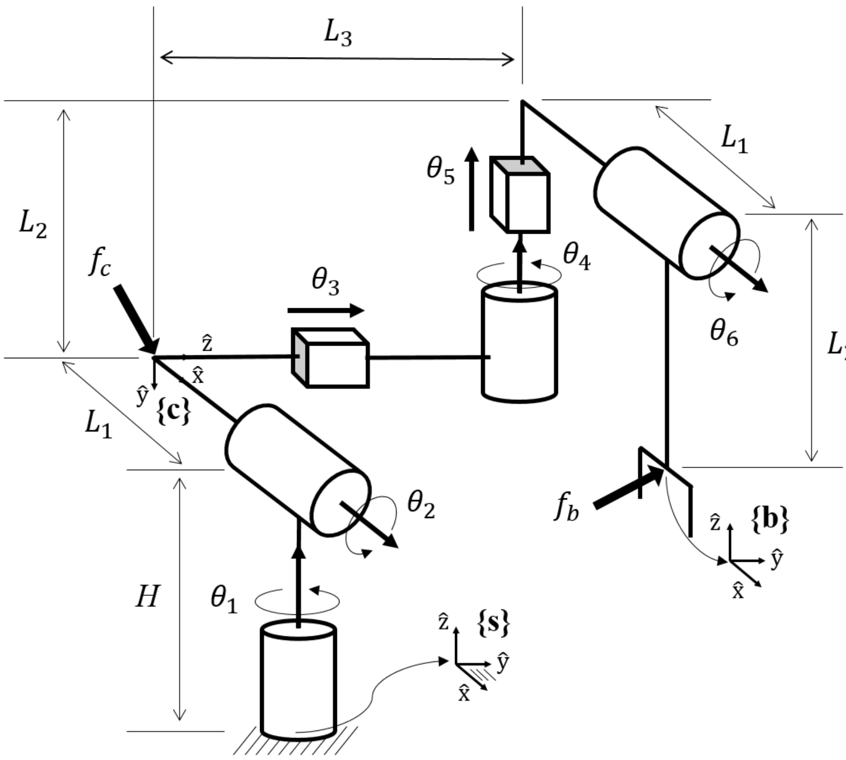


Figure 3: RRPRPR arm for Problem 3 shown in its zero position

Problem 4 (45 points)
Figure 4 shows a ball-catching robot arm, together with a camera used to track the ball motion. Let {s} and {c} be fixed frames attached respectively the robot base and camera, and {o} be a moving frame attached to the ball. Suppose T_{sc} and T_{co} are given as follows:

$$T_{sc} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -4\sqrt{3} \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{co} = \begin{bmatrix} \cos\frac{\sqrt{5}}{2}\pi t & -\sin\frac{\sqrt{5}}{2}\pi t & 0 & -6 \\ \sin\frac{\sqrt{5}}{2}\pi t & \cos\frac{\sqrt{5}}{2}\pi t & 0 & 2\sqrt{15}t \\ 0 & 0 & 1 & -\frac{1}{2}gt^2 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where g in T_{co} denotes the gravitational constant.
(a) Derive the linear velocity $v_{ball} \in \mathbb{R}^3$ and angular velocity $\omega_{ball} \in \mathbb{R}^3$ of the ball frame {o}, where both v_{ball} and ω_{ball} are expressed in fixed frame {s} coordinates.
(b) The robot must catch the ball at $t = \frac{1}{\sqrt{5}}$. Find at least one inverse kinematic solution for the robot at $t = \frac{1}{\sqrt{5}}$. Assume $g = 10$ for this part.
(c) To minimize the contact force when the gripper catches the ball, the end-effector frame's linear velocity should exactly match that of the ball. Derive the required joint velocities $\dot{\theta}$ at contact. Be sure to specify which inverse kinematic solution you use for this part. If you were unable to find an inverse kinematic solution, then explain the procedure for finding $\dot{\theta}$ assuming some inverse kinematic solution.

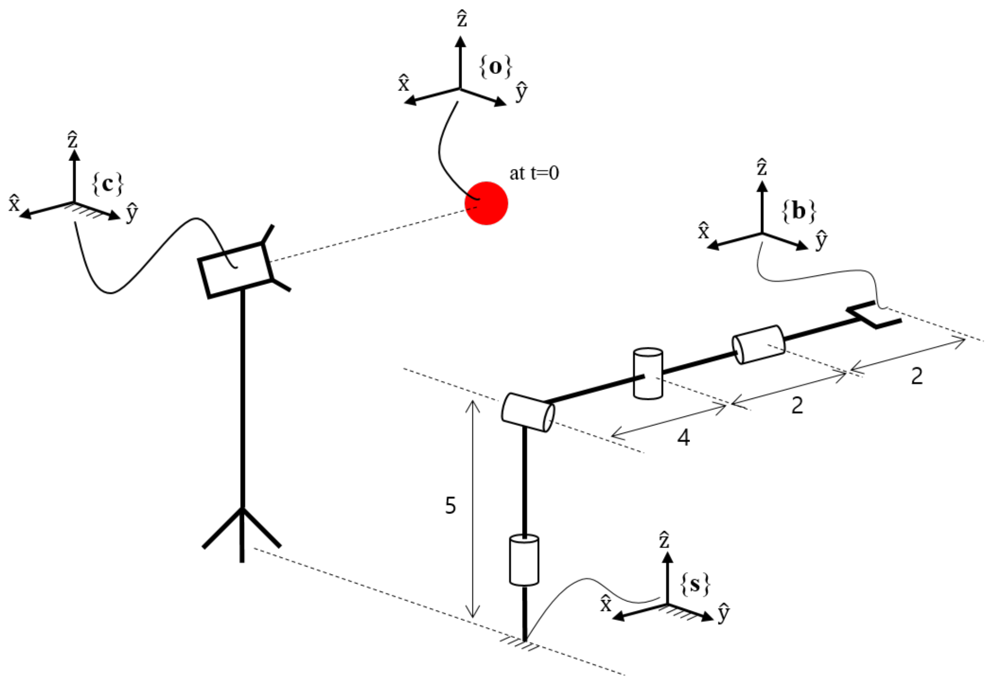


Figure 4: A ball-tracking camera and $4R$ robot (shown in its zero position)

Problem 5 (30 points)
In the Tower of Hanoi problem, the goal is to move the three rings shown in Figure 5(a) to the configuration shown in Figure 5(c) according to the following rules:

1. Only one ring can be moved at a time.
2. Each move consists of taking the upper ring from one of the stacks and placing it on the top of another stack or on an empty rod.
3. No larger ring may be placed on top of a smaller ring.

Consider the problem of moving from Figure 5(a) to Figure 5(c).
(a) Use Dijkstra's algorithm to find a path from Figure 5(b) to Figure 5(c). (**Note**: Setting the heuristic cost to zero in the A^* algorithm leads to Dijkstra's algorithm.)
(b) Use the A^* algorithm to find a path from Figure 5(a) to Figure 5(b). Define an appropriate heuristic cost, and at each iteration show the configuration of the three disks, and what is in the closed set and open set. Pseudo-code for the A^* algorithm is given in Figure 6.

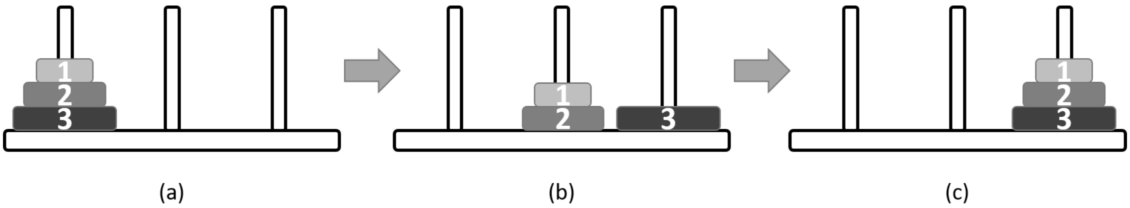


Figure 5: Tower of Hanoi problem for three rings

Algorithm 10.1 A^* search.

```
OPEN ← {1}
past_cost[1] ← 0, past_cost[node] ← infinity for node ∈ {2, ..., N}
while OPEN is not empty do
  current ← first node in OPEN, remove from OPEN
  add current to CLOSED
  if current is in the goal set then
    return SUCCESS and the path to current
  end if
  for each nbr of current not in CLOSED do
    tentative_past_cost ← past_cost[current]+cost[current,nbr]
    if tentative_past_cost < past_cost[nbr] then
      past_cost[nbr] ← tentative_past_cost
      parent[nbr] ← current
      put (or move) nbr in sorted list OPEN according to
          est_total_cost[nbr] ← past_cost[nbr] +
                   heuristic_cost_to_go(nbr)
    end if
  end for
end while
return FAILURE
```

Figure 6: A^* algorithm

Problem 6 (45 points)

(a) A point robot moves in $\mathbb{R}^2$ with coordinates $x \in \mathbb{R}^2$. Let the origin (0,0) be the goal, and suppose a circular obstacle of radius 2 is placed at $z = (0, -4)$. The following potential function is used to plan a path:

$$f(x) = \frac{1}{2}x^T Q x - \alpha(x-z)^T(x-z),$$

where α is a positive scalar and

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & \lambda_{\max} \end{bmatrix}. \tag{1}$$

Assuming the robot starts at (10, 0), set $\lambda_{\max} = 3$, $\alpha = 1$, and perform two iterations of gradient descent with fixed stepsize $h = 0.5$. Do you think a collision-free path to the goal can be found after enough iterations? If not, explain why not, and suggest some ways to fix the problem.

(b) The point robot now moves on the ellipsoid defined by $x^T Q x = 9$, where Q is as defined in (1) with $\lambda_{\max} = 3$. Its goal is to get as close as possible to the line defined by $x_1 + x_2 = 6$. Formulate an optimization problem and write down the first-order necessary conditions. Show that at if $x_e^* \in \mathbb{R}^2$ and $x_l^* \in \mathbb{R}^2$ are the respective points on the ellipsoid and line that are closest to each other, then $x_e^* - x_l^*$ is orthogonal to the line.

(c) The point robot now moves in a complicated obstacle-filled environment that can be represented by the graph of Figure 7. Use dynamic programming to find the shortest path from node 1 to node 12 in Figure 7, **moving only to the right**.

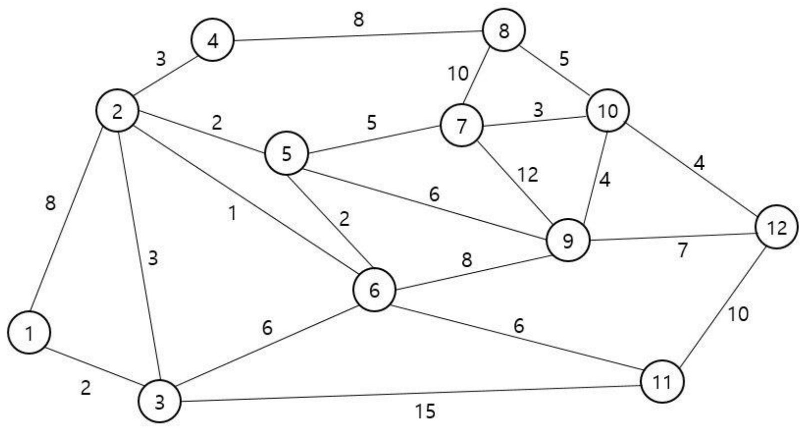


Figure 7: Path planning graph for Problem 6(c)

M2794.0027 Introduction to Robotics
Final Examination Solutions
June 11, 2019

Problem 1

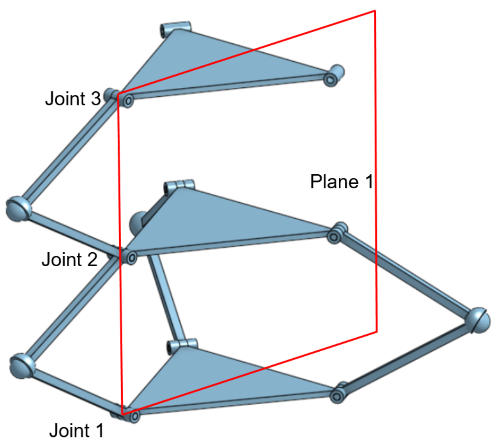
(a) Apply the spatial version of Grübler's formula as follows:

- $N = 1$ (ground) $+ 1$ (movable triangle) $+ 6$ (links) $= 8$;
- $J = 6$ (R joints) $+ 3$ (S joints) $= 9$;
- $\sum f_i = 1 \times 6$ (R joints) $+ 3 \times 3$ (S joints) $= 15$;
- $\text{DoF} = 6(N - 1 - J) + \sum f_i = 6(8 - 1 - 9) + 15 = 3.$

(b) Again apply the spatial version of Grübler's formula:

- $N = 1$ (ground) $+ 1$ (movable triangle) $+ 12$ (links) $= 14$;
- $J = 12$ (R joints) $+ 6$ (S joints) $= 18$;
- $\sum f_i = 1 \times 12$ (R joints) $+ 3 \times 6$ (S joints) $= 30$;
- $\text{DoF} = 6(N - 1 - J) + \sum f_i = 6(14 - 1 - 18) + 30 = 0.$

Note that the top and lower triangles are fixed to ground; it remains to determine whether the middle triangle can in fact move, the results of Grübler's formula notwithstanding. To answer this question, consider Figure 1(a), in which only one of the three leg structures is drawn. In this case joints 1, 2, and 3 are parallel and thus admit link motions in the plane drawn. A similar analysis holds for the remaining two leg structures, leading to the conclusion that the middle triangle can in fact translate vertically along the $\hat{z}_0$-axis with one degree of freedom (represented as the intersection of three planes as shown in Figure 1(b)). Although Grübler's formula predicts zero degrees of freedom, the inherent symmetry of the design in fact allows for one degree of translational freedom.



(a)

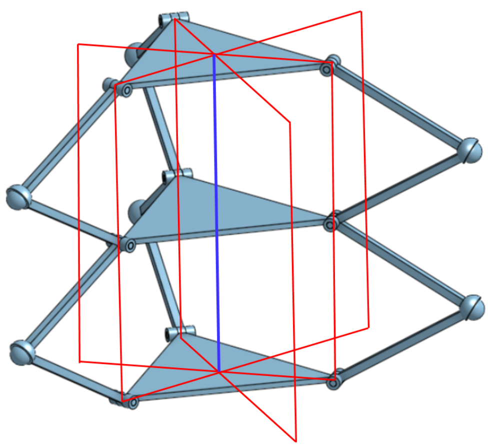

(b)

Figure 1: Figures for Problem 1.

(c) Given a rotation axis $\hat{\omega}$ and angle θ, the corresponding rotation is given by

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} \quad (||\hat{\omega}|| = 1, \theta \in [0, \pi]).$$

Letting $\mathrm{Rot}(\hat{z}_1, \varphi) = R(\varphi)$,

$$\mathrm{Rot}(\hat{z}_1, \varphi)\mathrm{Rot}(\hat{y}_1, 2\psi)\mathrm{Rot}(\hat{z}_1, -\varphi) = R(\varphi)e^{[\hat{y}_1]2\psi}R(\varphi)^\top = e^{R(\varphi)[\hat{y}_1]R^\top 2\psi} = e^{[R(\varphi)\hat{y}_1]2\psi}.$$

Since $R(\varphi)\hat{y}_1$ is confined to be a unit vector in the $\hat{x}_0$-$\hat{y}_0$ plane and $\hat{\omega} = \frac{1}{\sqrt{3}}(1, 1, 1)^\top$, it follows that

$$R(\varphi)\hat{y}_1 \neq \hat{\omega}$$

for any φ. The 3-RSR mechanism therefore cannot generate the given orientation.

Problem 2

(a) Express the static equilibrium force closure conditions in the standard linear form $Ax = b$, where $A \in \mathbb{R}^{3\times4}$ is given, and the objective is to determine whether a nonnegative solution $x \geq 0$ exists for any arbitrary $b \in \mathbb{R}^3$. Setting up the problem in this way leads to Gauss-Jordan elimination of the following matrix:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 1 \\ -1 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Force closure requires that all entries of the fourth column be negative. Since the third entry of the fourth column is not negative, the grasp is not force closure.

(b) Place the reference frame at the intersection of the lines of action at A and B, and define x^* to be the distance along the $\hat{x}$-direction between the origin of the reference frame and the lower-left corner of the triangle. Figure 2 illustrates the situation. Now express the static equilibrium force closure conditions in the standard linear form $Ax = b$, where $A \in \mathbb{R}^{3\times4}$ is given, and the objective is to determine whether a nonnegative solution $x \geq 0$ exists for any arbitrary $b \in \mathbb{R}^3$. Setting up the problem in this way leads to Gauss-Jordan elimination of the following matrix:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 & 1 \\ 0 & 0 & x_3 - x^* & x_4 - x^* \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 + \frac{x_4 - x^*}{x_3 - x^*} \\ 0 & 1 & 0 & -1 + \frac{x_4 - x^*}{x_3 - x^*} \\ 0 & 0 & 1 & \frac{x_4 - x^*}{x_3 - x^*} \end{bmatrix}.$$

Force closure requires that all entries of the fourth column be negative. Simplifying the corresponding inequalities leads to the following condition:

$$\frac{x_4 - x^*}{x_3 - x^*} < 0,$$

which leads to one of two possibilities (i) $x_4 - x^* < 0$ and $x_3 - x^* > 0$; (ii) $x_4 - x^* > 0$ and $x_3 - x^* < 0$. The above asserts that the lines of action at A and B must intersect at a point that lies between the lines of action at C and D.

(c) Assume point contacts A and B are frictionless, while C is a frictional point contact with $\mu(\equiv \tan\alpha > 0)$; Figure 3 illustrates the grasp. Now express the static equilibrium force closure conditions in the standard linear form $Ax = b$, where $A \in \mathbb{R}^{3\times4}$ is given and the objective is to determine whether a nonnegative solution $x \geq 0$ exists for any arbitrary $b \in \mathbb{R}^3$; setting up the problem in this way leads to Gauss-Jordan elimination of the following matrix:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & -\sin\alpha & \sin\alpha \\ -\frac{1}{2} & -\frac{1}{2} & \cos\alpha & \cos\alpha \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & -2\cos\alpha \\ 0 & 1 & 0 & -2\cos\alpha \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Force closure requires that all entries of the fourth column be negative, i.e., $\cos\alpha > 0$, which is satisfied for any $0 < \alpha < \frac{\pi}{2}$, or equivalently, $0 < \mu$.

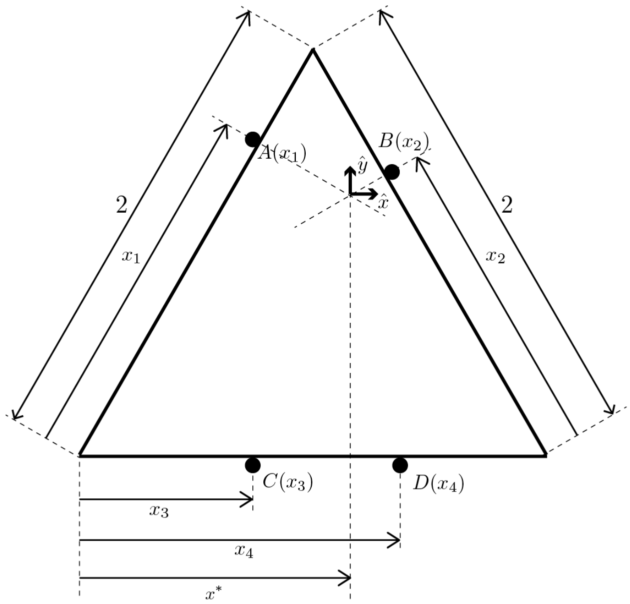


Figure 2: Figure for Problem 2(b).

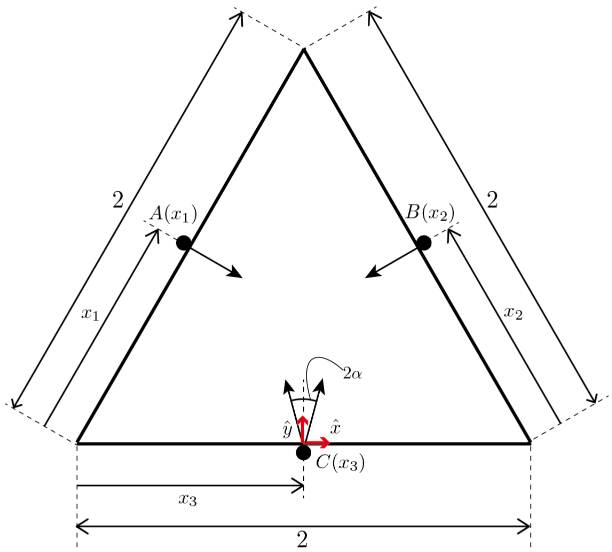


Figure 3: Figure for Problem 2(c).

Problem 3

(a) The requirement that $\alpha_i \in [0, \pi]$, leads to two possible solution sets for $(\alpha_{i-1}, a_{i-1}, d_i, \phi_i)$ (depending on the choice of direction for $\hat{x}_3$):

i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	H	$90° + \theta_1$
2	$90°$	0	$-L_1$	$90° + \theta_2$
3	$90°$	0	$L_3 + \theta_3$	$\pm 90°*$

The corresponding link frames are shown in Figure 4.

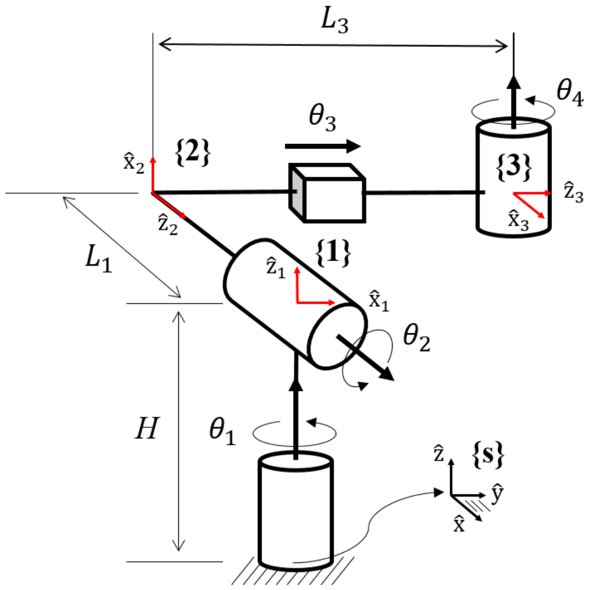


Figure 4: Corresponding link frames for (a) ($\theta_3 = 90°$).

(b) T_{sb} can be expressed in the following PoE form:

$$T_{sb} = \left(e^{[\mathcal{A}_1]\theta_1} e^{[\mathcal{A}_2]\theta_2} M_{sc}\right)\left(e^{[\mathcal{A}_3]\theta_3} e^{[\mathcal{A}_4]\theta_4} M_{cb} e^{[\mathcal{A}_5]\theta_5} e^{[\mathcal{A}_6]\theta_6}\right) = T_{sc}T_{cb},$$

where M_{ij} is the transformation from frame $\{i\}$ to frame $\{j\}$ when the robot is in its zero position. From this interpretation it follows that $\mathcal{A}_1$ are $\mathcal{A}_2$ are the joint screw vectors for joints 1 and 2 expressed in frame $\{s\}$, while $\mathcal{A}_3$ is the joint screw vector for joint 3 expressed in frame $\{c\}$. M_{sc}, M_{cb}, and $\mathcal{A}_i = (\hat{\omega}_i, v_i)$, $i = 1, 2, 3$ are as follows:

$$M_{sc} = \begin{bmatrix} 1 & 0 & 0 & -L_1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & H \\ 0 & 0 & 0 & 1 \end{bmatrix}, M_{cb} = \begin{bmatrix} 1 & 0 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i	$\hat{\omega}_i$	q_i	v_i
1	$(0, 0, 1)$	$(0, 0, 0)$	$(0, 0, 0)$
2	$(1, 0, 0)$	$(0, 0, H)$	$(0, H, 0)$
3	$(0, 0, 0)$	-	$(0, 0, 1)$

(c) In order to keep the robot in static equilibrium, the following wrenches need to be generated at points C and B:

$$\mathcal{F}_{c,gen} = \begin{bmatrix} 0 \\ -f_c \end{bmatrix} = (0, 0, 0, -1, -1, -1)^T, \quad \mathcal{F}_{b,gen} = \begin{bmatrix} 0 \\ -f_b \end{bmatrix} = (0, 0, 0, -1, 1, 1)^T$$

Let J_b be the manipulator body Jacobian, and J_c be the Jacobian for frame $\{c\}$ taking into account only joints 1 and 2 (i.e., $J_c \in \mathbb{R}^{6\times 2}$). The required joint torques $\tau_b \in \mathbb{R}^6$, $\tau_c \in \mathbb{R}^2$ are then obtained as $\tau_b = {J_b}^T \mathcal{F}_{b,gen}$ and $\tau_c = {J_c}^T \mathcal{F}_{c,gen}$. J_b and J_c at the zero position can be obtained from the following joint screw parameters:

i	$\hat{\omega}_i$	q_i	v_i
1	$(0,0,1)$	$(0,-L_3,0)$	$(-L_3,0,0)$
2	$(1,0,0)$	$(0,-L_3,0)$	$(0,0,L_3)$
3	$(0,0,0)$	-	$(0,1,0)$
4	$(0,0,1)$	$(-L_1,0,0)$	$(0,L_1,0)$
5	$(0,0,0)$	-	$(0,0,1)$
6	$(1,0,0)$	$(0,0,L_2)$	$(0,L_2,0)$

$$\Rightarrow \quad J_b = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -L_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L_1 & 0 & L_2 \\ 0 & L_3 & 0 & 0 & 1 & 0 \end{bmatrix},$$

i	$\hat{\omega}_i$	q_i	v_i
1	$(0,-1,0)$	$(L_1,0,0)$	$(0,0,-L_1)$
2	$(1,0,0)$	$(0,0,0)$	$(0,0,0)$

$$\Rightarrow \quad J_c = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -L_1 & 0 \end{bmatrix}.$$

The total required joint torque τ_{tot} is then

$$\begin{aligned} \tau_{tot} &= \tau_b + \begin{bmatrix} \tau_c \\ \mathbf{0}^{4\times 1} \end{bmatrix} = {J_b}^T \mathcal{F}_{b,gen} + \begin{bmatrix} {J_c}^T \mathcal{F}_{c,gen} \\ \mathbf{0}^{4\times 1} \end{bmatrix} \\ &= \begin{bmatrix} L_3 \\ L_3 \\ 1 \\ L_1 \\ 1 \\ L_2 \end{bmatrix} + \begin{bmatrix} L_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} L_1 + L_3 \\ L_3 \\ 1 \\ L_1 \\ 1 \\ L_2 \end{bmatrix}. \end{aligned}$$

Problem 4

(a) Straightforwardly,

$$[\mathcal{V}_o] = T_{co}^{-1}\dot{T}_{co} = \begin{bmatrix} 0 & -\frac{\sqrt{5}}{2}\pi & 0 & 2\sqrt{15}\sin\frac{\sqrt{5}}{2}\pi t \\ \frac{\sqrt{5}}{2}\pi & 0 & 0 & 2\sqrt{15}\cos\frac{\sqrt{5}}{2}\pi t \\ 0 & 0 & 0 & -gt \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which can be expressed in twist vector form $\mathcal{V}_o = (\omega_o, v_o)$ with

$$\omega_o = (0, 0, \frac{\sqrt{5}}{2}\pi)^T, \quad v_o = (2\sqrt{15}\sin\frac{\sqrt{5}}{2}\pi t, 2\sqrt{15}\cos\frac{\sqrt{5}}{2}\pi t, -gt)^T.$$

Since w_o and v_o are respectively the angular and linear velocities of frame {o} expressed in frame {o}, the corresponding angular and linear velocities of frame {o} expressed in frame {s} can be obtained via multiplication by R_{so}:

$$\omega_s = R_{so}\omega_o = \begin{bmatrix} 0 \\ 0 \\ \frac{\sqrt{5}}{2}\pi \end{bmatrix}, \quad v_s = R_{so}v_o = \begin{bmatrix} 0 \\ 2\sqrt{15} \\ -gt \end{bmatrix}.$$

(b) When $t = \frac{1}{\sqrt{5}}$,

$$T_{so} = \begin{bmatrix} 0 & 1 & 0 & -6 \\ -1 & 0 & 0 & -2\sqrt{3} \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Seen from the plane $z = 5$ and assuming $\theta_2 = 0$, the elbow-down solution can be obtained via inspection as

$$\theta_{\text{sol}} = (0, 0, \pi/3, x)^T,$$

where x any arbitrary joint angle. Three more solutions exist (one of these is shown in Figure 5); specifying any of these four solutions is sufficient.

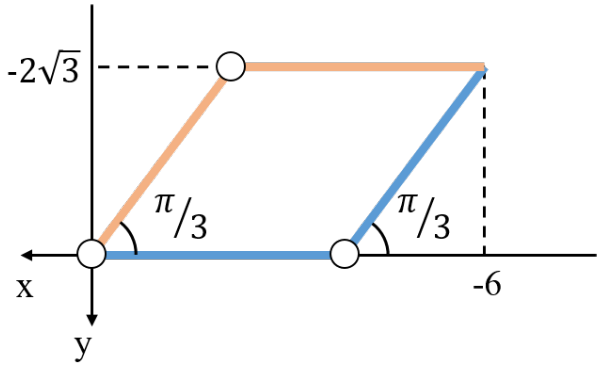


Figure 5: Inverse kinematics solutions

(c) From $\mathcal{V}_o$ obtained in (a), the linear velocity of frame {o} expressed in frame {o} coordinates is $(2\sqrt{15}, 0, -2\sqrt{5})$. For the inverse kinematics configuration described in part (b), the linear velocity of frame {b} expressed in frame {b} coordinates is obtained as

$$R_{bo} \begin{bmatrix} 2\sqrt{15} \\ 0 \\ -2\sqrt{5} \end{bmatrix} = \text{Rot}(\hat{z}, \pi/6) \begin{bmatrix} 3\sqrt{5} \\ \sqrt{15} \\ -2\sqrt{5} \end{bmatrix},$$

since frame {o}'s position and linear velocity matches that of frame {b} when $t = \frac{1}{\sqrt{5}}$. The desired joint velocities can then be obtained using the body Jacobian:

$$J_b(\theta_{\text{sol}}) = \begin{bmatrix} 0 & \frac{1}{2}\sqrt{3} & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -2\sqrt{3} & 0 & 0 & 0 \\ -6 & 0 & -4 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

Using only the lower three rows of the body Jacobian for the linear velocity, we solve the following equation for $\dot{\theta}$:

$$\begin{bmatrix} 3\sqrt{5} \\ \sqrt{15} \\ -2\sqrt{5} \end{bmatrix} = \begin{bmatrix} -2\sqrt{3} & 0 & 0 & 0 \\ -6 & 0 & -4 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix},$$

leading to

$$\dot{\theta} = \begin{bmatrix} -\frac{1}{2}\sqrt{15} \\ -\frac{1}{3}\sqrt{5} \\ \frac{1}{2}\sqrt{15} \\ x \end{bmatrix}$$

for x any arbitrary value. For the elbow-up case shown in Figure 5, using the same procedure we obtain

$$\dot{\theta} = \begin{bmatrix} 0 \\ -\frac{1}{3}\sqrt{5} \\ -\frac{1}{2}\sqrt{15} \\ x \end{bmatrix}.$$

Problem 5

(a) Each configuration for the three-disk, three-rod Tower of Hanoi problem is represented by a 3×3 table. Recall that Dijkstra's algorithm can be obtained from the A^* algorithm by setting the heuristic cost h to zero, so that $f = g$. The result of Dijkstra's algorithm is given in Figure 6. The final path is $\{1, 2, 4, 6\}$ as shown in Figure 6.

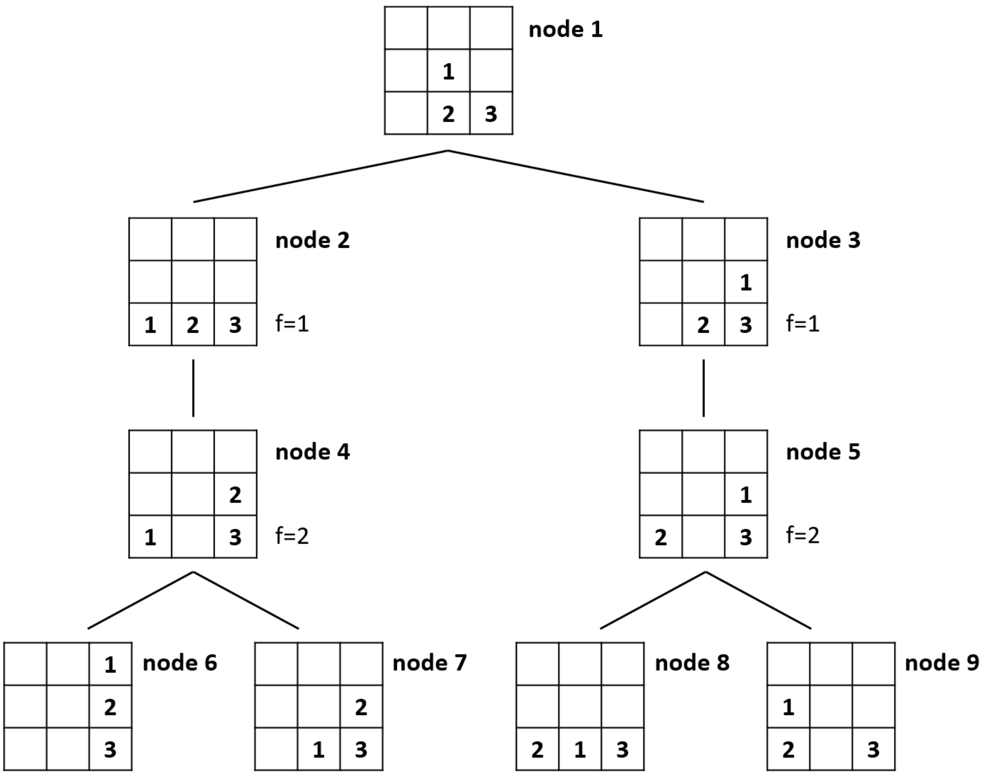


Figure 6: Figures for Problem 5(a).

(b) To find the optimal path using A^*, recall that the heuristic cost should be an underestimate of the actual cost. One example of such a heuristic cost is to count the number of moves needed to place disk 3 at the bottom of the third rod in the following way: (number of disks on top of disk 3) + (number of disks on the third rod) + 1 (the latter 1 corresponding to moving disk 3 to the bottom of the third rod). This heuristic underestimates the true cost, since it assumes that disks 1 and 2 can always be placed on rod 2 in no more than 2 steps. Figure 7 illustrates the result for finding the path using this heuristic cost.

Iteration 1: CLOSED = $\{1\}$, OPEN = $\{2, 3\}$

Iteration 2: CLOSED = $\{1, 2\}$, OPEN = $\{4, 3\}$

Iteration 3: CLOSED = $\{1, 2, 4\}$, OPEN = $\{3, 5, 6\}$

Iteration 4: CLOSED = $\{1, 2, 4, 3\}$, OPEN = $\{7, 5, 6\}$

Iteration 5: CLOSED = $\{1, 2, 4, 3, 7\}$, OPEN = $\{9, 8, 5, 6\}$

Iteration 6: CLOSED = $\{1, 2, 4, 3, 7, 9\}$, OPEN = $\{10, 8, 5, 6\}$

Iteration 7: CLOSED = $\{1, 2, 4, 3, 7, 9, 10\}$, OPEN = $\{8, 5, 6\}$.

The final path is $\{1, 3, 7, 9, 10\}$.

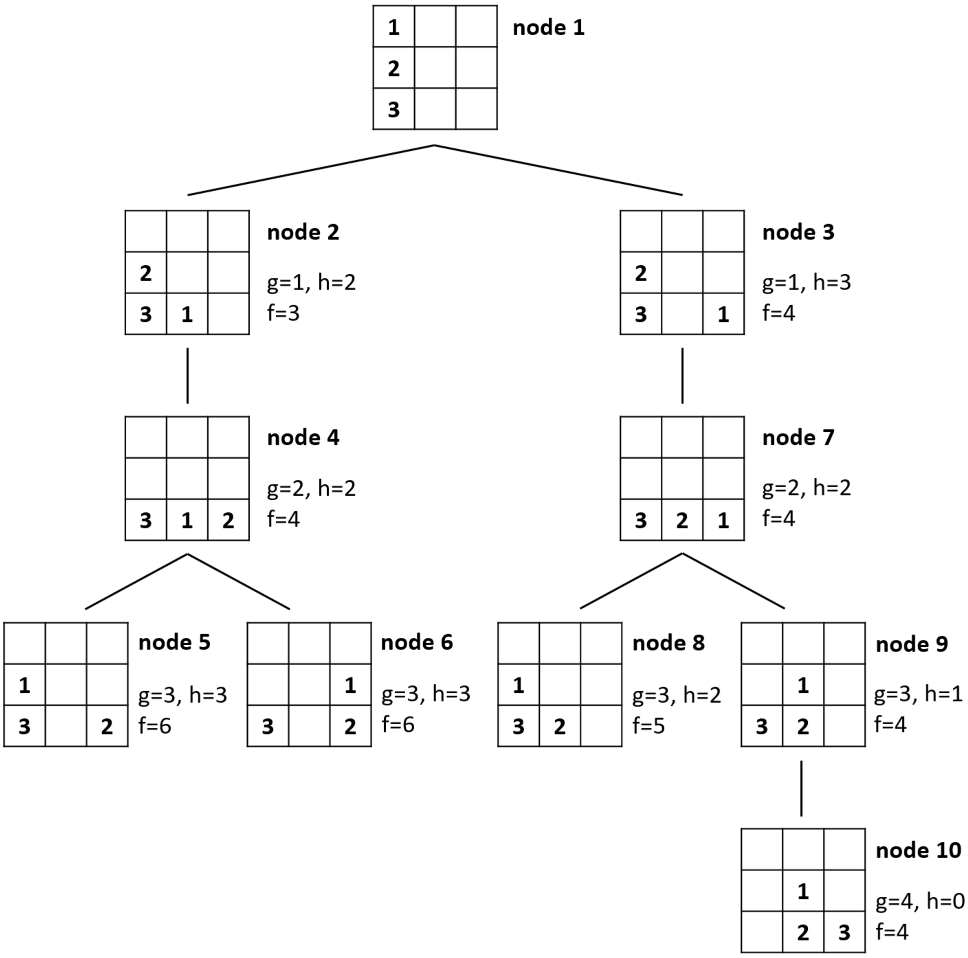


Figure 7: Figure for Problem 5(b)

Problem 6

(a) The gradient is given by

$$\frac{\partial f}{\partial x} = x^T Q + (-2\alpha(x-z)^T).$$

The first thing to note is that the critical point of $f(x)$, i.e., the solution to the first-order necessary condition $\frac{\partial f}{\partial x} = x^T Q + (-2\alpha(x-z)^T) = 0$, is given uniquely by (assuming the matrix inverse exists)

$$x^* = -2\alpha(Q - 2\alpha I)^{-1} z,$$

which is **not** the goal configuration z. The potential method will therefore drive the robot to the incorrect goal configuration. This is indeed evident from two iterations of gradient descent with parameters $\lambda_{\max} = 3$, $\alpha = 1$, and $h = 0.5$ as prescribed in the problem statement:

$$\begin{aligned}
\frac{\partial f}{\partial x}(x_0) &= \begin{bmatrix} 10 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} - 2 \begin{bmatrix} 10 & 4 \end{bmatrix} = \begin{bmatrix} -10 & -8 \end{bmatrix} \\
x_1 &= x_0 - 0.5 \begin{bmatrix} -10 & -8 \end{bmatrix}^T = \begin{bmatrix} 15 \\ 4 \end{bmatrix} \\
\frac{\partial f}{\partial x}(x_1) &= \begin{bmatrix} 15 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} - 2 \begin{bmatrix} 15 & 8 \end{bmatrix} = \begin{bmatrix} -15 & -4 \end{bmatrix} \\
x_2 &= x_1 - 0.5 \begin{bmatrix} -15 & -4 \end{bmatrix}^T = \begin{bmatrix} 22.5 \\ 6 \end{bmatrix}.
\end{aligned}$$

Reducing the effect of the repulsive potential term when the robot is far away from the obstacle is also helpful, e.g., by defining the repulsive potential to be zero for configurations beyond a fixed distance from an obstacle boundary, and making the repulsive function sharper as the robot approaches the obstacle boundary; a common obstacle potential function discussed in class, for example, is

$$P_{obs} = \frac{\alpha}{(x-z)^T(x-z)}.$$

The overall shape of the potential function can be adjusted by varying the parameters $\lambda_{\max}$ and α. Also, the stepsize h need not be fixed; one could, for example, choose h that results in the maximum decrease of the potential function along the given search direction (so-called steepest descent), although this method also has its own inherent problems, e.g., slow convergence in long and narrow valleys, zig-zag paths, etc.

Many students made the mistake of trying to apply the Newton-Raphson root-finding algorithm to the potential function rather than directly minimizing the potential; although the algorithms may superficially resemble each other, note that the potential function is a mapping from $\mathbb{R}^2$ to $\mathbb{R}$, whereas Newton-Raphson is intended to find roots of mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ between spaces of the same dimension. One could in principle try to solve the first-order necessary conditions using Newton-Raphson, but in this case second derivatives of the potential function are needed (this method corresponds to what is known as Newton's method in optimization, and can be viewed as taking local second-order approximations of the potential function).

(b) The corresponding optimization problem can be formulated as the following equality-constrained minimization:

$$\begin{aligned}
\underset{x_e, x_l}{\text{minimize}} \quad & \frac{1}{2}\|x_e - x_l\|^2 \\
\text{subject to} \quad & x_e^T \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} x_e = 9 \,, x_l^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 6.
\end{aligned}$$

Setting up the first-order necessary conditions,

$$L(x_e, x_l, \lambda_e, \lambda_l) = \frac{1}{2}(x_e - x_l)^T(x_e - x_l) + \lambda_e(9 - x_e^T Q x_e) + \lambda_l(6 - x_l^T)\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

and

$$\frac{\partial L}{\partial x_e} = (x_e - x_l)^T + \lambda_e(-2x_e^T Q) = 0 \quad (1)$$

$$\frac{\partial L}{\partial x_l} = -(x_e - x_l)^T + \lambda_l \begin{bmatrix} -1 & -1 \end{bmatrix} = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda_e} = 9 - x_e^T Q x_e = 0 \quad (3)$$

$$\frac{\partial L}{\partial \lambda_l} = 6 - x_l^T = 0. \quad (4)$$

From (2) we obtain $(x_e - x_l)^T = (\lambda_l, \lambda_l)$, and can conclude that $x_e - x_l$ is orthogonal to the line $x_1 + x_2 = 6$.
(c) Using dynamic programming, the optimal path is $1 \to 2 \to 5 \to 7 \to 10 \to 12$.

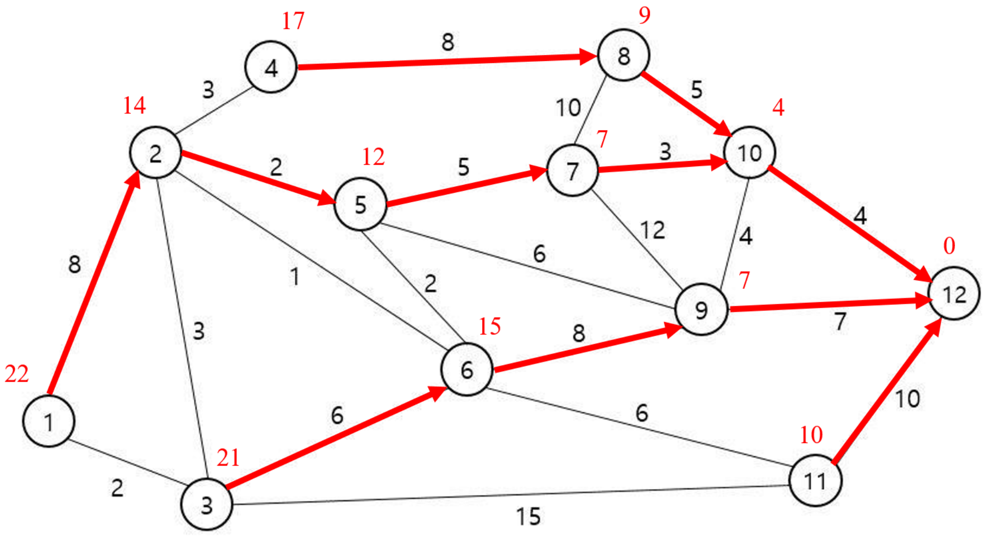


Figure 8: Figure for Problem 6(c)