ISSN 0025-6544, Mechanics of Solids, 2010, Vol. 45, No. 1, pp. 27–33. © Allerton Press, Inc., 2010.
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Brachistochrone for a Rolling Cylinder

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Received January 17, 2008

Abstract—The motion of a heavy homogeneous cylinder is considered as a no-slip rolling along the desired curve. We obtain a functional in the form of the total time of the cylinder rolling and solve the corresponding variational problem of minimizing this functional. We obtain an algebraic equation for the directional line of steepest descent, brachistochrone, in parametric form. We use the equation of motion of the cylinder with constraint reaction to determine the conditions of implementation of its pure rolling without separation and slip with respect to the brachistochrone.

DOI: 10.3103/S002565441001005X

Key words: *brachistochrone, cylinder rolling without slip, variational problem, functional minimization, cycloid, Euler–Poisson equation.*

1. INTRODUCTION

In 1696, Johann Bernoulli posed the following brachistochrone problem: find the shape of the curve along which a bead that is at rest at the initial time and is accelerated by gravity descents from one given point to the other given point in the least time interval. In this problem, it is assumed that the material point (bead) moves in the vertical plane and in a homogeneous gravitational field.

I. Newton, G. Leibniz, G. L'Hospital, Jakob Bernoulli, and Johann Bernoulli showed that the solution of this problem is a cycloid [1].

Analytic solutions of the brachistochrone problem based on the use of the classical technique of calculus of variations are given in [2], and the analytic solutions in the case of geometric optics are given in [3].

The problems of determining the brachistochrone shape with Coulomb friction taken into account in the motion of a material point in a vertical plane under the action of a homogeneous field of gravity were studied in [4–6].

Generalizations of the problem of searching the brachistochrone shape on the cylinder with Coulomb friction taken into account can be found in [7]. The corresponding problem of determining the brachistochrone shape in nonconservative force fields was considered in [8]. The problem of determining the brachistochrone on the cylinder in homogeneous force fields was solved in [9], and on cylinders and on the sphere, in the unpublished paper [10]. Generalizations of this problem to inhomogeneous fields were considered in [11–13], and the brachistochrone problem in linear radial force fields was solved in [13]. Solutions of the same problem in radial forces with the force dependence inversely proportional to the squared distance between the interacting points were found in [12, 14, 15].

Further generalizations of the problem on a brachistochrone with a material point moving on it, including relativistic effects, can be found in [16, 17].

Problems on a brachistochrone with finite-dimensional bodies rolling on it are considered in [18, 19].

The present paper is motivated by the following technical problem. For many vibroprotective systems, dampers, and stabilizers, the main criterion of efficiency of their operation is the minimum of displacements of certain points of the carrying object or the minimum of forces (moments) in the most dangerous cross-sections, which arise in the process of its forced vibrations. But there are several vibroprotective devices for which the main criterion of their operation efficiency is the minimum of time in which the level of amplitudes of the bearing structure forced oscillations can be reduced to an admissible level [20–22].

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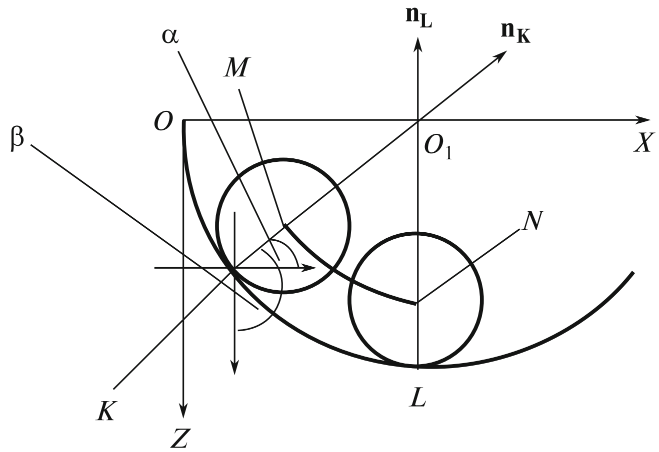

Fig. 1.

In the present paper, such a criterion is used for vibroprotective devices of roller type [23, 24]. In this connection, there is a problem of determining the shape of the directional curve of the cylindrical surface along which the heavy cylinder descents in minimum time. In this case, the cylinder motion along the curve is a no-slip rolling, and the cylinder itself is assumed to be homogeneous in the present paper. The no-slip rolling is related to technical requirements of vibroprotection problems.

2. STATEMENT OF THE PROBLEM AND CONSTRUCTION OF THE OBJECTIVE FUNCTIONAL

We assume that a homogeneous cylinder of mass m and radius r is in a homogeneous field of gravity and begins its motion from point O without initial velocity rolling without slip along a certain cylindrical cavity with directional curve OKL to point L (see Fig. 1). In this case, curve OKL lies in the vertical plane. The cylinder center is at point M.

The problem is to find curve OKL such that moving along this curve the cylinder rolls from point O to point L in the least possible time.

To determine the shape of curve OKL, we introduce the rectilinear Cartesian coordinate system OXZ with origin at point O and direct the axis OX horizontally to the right and the axis OZ vertically downwards. We assume that the equation of the desired curve OKL is described by the function $z = z(x)$. Thus, the cylinder motion occurs in the plane OXZ.

We determine the expression for the time interval necessary for the cylinder to move from the initial point $O(0; 0)$ on the curve to some given point $L(x_L; z(x_L))$ on the plane OXZ.

We write the vector of the unit interior normal to the desired curve at its arbitrary (current) point K:

$$\mathbf{n}_K = (\cos\alpha; \cos\beta) = \left(\frac{z'}{\sqrt{1+(z')^2}}; -\frac{1}{\sqrt{1+(z')^2}}\right), \tag{2.1}$$

where α is the angle between the vector $\mathbf{n}_K$ and the positive direction of the axis OX, β is the angle between the vector $\mathbf{n}_K$ and the positive direction of the axis OZ, and $z' = dz/dx$.

Then the expression for the cylinder potential energy has the form

$$\Pi = -mg(z + r\cos\beta) = -mg\left(z - \frac{r}{\sqrt{1+(z')^2}}\right) + C_0, \tag{2.2}$$

where $r = KM = LM$ is the cylinder radius.

With (2.2) taken into account, we write out the expression for the energy integral in this problem:

$$\frac{mv^2}{2} + \frac{J\dot{\psi}^2}{2} = mg\left(z - \frac{r}{\sqrt{1+(z')^2}}\right) - C_0, \tag{2.3}$$

where $J = \frac{1}{2}mr^2$ is the moment of inertia of a homogeneous cylinder with respect to the horizontal axis passing through its center of mass and $v = r\dot{\psi}$ is the velocity of the cylinder center of mass written in terms of its angular velocity $\dot{\psi}$ in the case of no-slip motion of the cylinder.

Since, according to the imposed initial conditions $z(0) = 0$, $z'(0) = z'_0$, the velocity of the cylinder center of mass is zero $v = 0$, it follows that the constant is $C_0 = -mgr/\sqrt{1 + (z'_0)^2}$ and the energy integral (2.3) becomes

$$\frac{3m(r\dot{\psi})^2}{4} = mg\left(z - \frac{r}{\sqrt{1 + (z')^2}} + \bar{C}\right), \quad \bar{C} = \frac{r}{\sqrt{1 + (z'_0)^2}}. \tag{2.4}$$

We use (2.4) to write out the expression for the time differential dt:

$$dt = \sqrt{\frac{3}{4g}} \frac{\sqrt{[1 + (z')^2]^3} + rz''}{[1 + (z')^2]\sqrt{z - r/\sqrt{1 + (z')^2} + \bar{C}}} dx. \tag{2.5}$$

The expression for the time interval on which the cylinder rolls along the curve OKL from point O to point L can be obtained by integrating the differential expression (2.5):

$$T[z(x)] = \sqrt{\frac{3}{4g}} \int_0^{x_L} \frac{\sqrt{[1 + (z')^2]^3} + rz''}{[1 + (z')^2]\sqrt{z - r/\sqrt{1 + (z')^2} + \bar{C}}} dx, \quad z(0) = 0, \quad z(x_L) = z_L. \tag{2.6}$$

Thus, expression (2.6) for time T is given by the functional of the form

$$T[z(x)] = \int_0^X F(z, z', z'')\, dx, \tag{2.7}$$

in which the integrand does not explicitly depend on x and the highest-order derivative $z''(x)$ occurs linearly. It is known [25] that for $F = F(z, z', z'')$ the Euler–Poisson equation

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\left(\frac{\partial F}{\partial z'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial z''}\right) = 0 \tag{2.8}$$

admits the first integral

$$F - \frac{\partial F}{\partial z'} z' - \frac{\partial F}{\partial z''} z'' + \left[\frac{d}{dx}\left(\frac{\partial F}{\partial z''}\right)\right] z' = C, \tag{2.9}$$

where C is an arbitrary constant.

Because the highest-order derivative $z''(x)$ occurs linearly in $F = F(z, z', z'')$, the order of the Euler–Poisson equation decreases. (The variational problem is degenerate.)

We substitute the expression $F = F(z, z', z'')$ from (2.6) into (2.9) and, after several cumbersome transformations, obtain the following differential equation for the desired function $z(x)$:

$$(z + \bar{C})[1 + (z')^2] - r\sqrt{1 + (z')^2} = C_1, \quad z(0) = 0, \quad z(x_L) = z_L, \tag{2.10}$$

where $C_1 = [C\sqrt{\frac{4}{3}g}]^{-2}$, g is the free fall acceleration, and r is the cylinder radius.

An important fact in Eq. (2.10) is that in the special case of $r = 0$ (the motion of a material point) it becomes the well-known cycloid equation [25, 26], namely, $(z + \bar{C})[1 + (z')^2] = C_1$.

3. INTEGRATION OF THE DIFFERENTIAL EQUATION OF THE DESIRED CURVE

We integrate Eq. (2.10) using a parametrization of the curve $z = z(x)$. The parametrization is introduced as follows:

$$z' = \cot\frac{\varphi}{2} \quad \Rightarrow \quad dx = \tan\frac{\varphi}{2}\, dz, \tag{3.1}$$

$$z = C_1 \sin^2 \frac{\varphi}{2} + r \sin \frac{\varphi}{2} - \bar{C} \quad \Rightarrow \quad dz = \left(C_1 \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} + \frac{1}{2} r \cos \frac{\varphi}{2} \right) d\varphi. \tag{3.2}$$

We substitute the right-hand side of (3.2) into (3.1) and obtain

$$dx = \tan \frac{\varphi}{2} \left(C_1 \cos \frac{\varphi}{2} \sin \frac{\varphi}{2} + \frac{1}{2} r \cos \frac{\varphi}{2} \right) d\varphi \quad \Rightarrow \quad dx = \left(C_1 \sin^2 \frac{\varphi}{2} + \frac{1}{2} r \sin \frac{\varphi}{2} \right) d\varphi. \tag{3.3}$$

After integration of (3.3), we obtain the expressions for x and z as a function of the parameter φ

$$x(\varphi) = \frac{C_1}{2}(\varphi - \sin\varphi) - r \cos \frac{\varphi}{2} + C_2, \tag{3.4}$$

$$z(\varphi) = \frac{C_1}{2}(1 - \cos\varphi) + r \sin \frac{\varphi}{2} - \bar{C}. \tag{3.5}$$

In Eqs. (3.4), (3.5), the constant $\bar{C}$ corresponds to the initial conditions of the cylinder motion (see formula (2.4)). The constants C_1 and C_2 are determined by the coordinates of the two points on the plane OXZ through which the optimal curve $z = z(x)$ must pass. Their number corresponds to the order of the Euler–Poisson equation. Let us find these constants.

We choose the initial position of the cylinder so that its center of mass be at the point with coordinates $(r; 0)$ and the point of tangency be at the point $O(0; 0)$. Then $\bar{C} = 0$. If $\bar{C} = r/\sqrt{1 + (z_0')^2} = r \sin(\varphi/2)\big|_{O(0;0)} \neq 0$, then from Eq. (3.5) we obtain $C_1 = 0$, and there is no solution. Thus, to the initial point $O(0; 0)$ in the rectangular Cartesian coordinate system there corresponds the initial parameter value $\varphi = 0$ in the parametric equations (3.4), (3.5).

We determine C_2 from Eq. (3.4). Since the desired curve $z = z(x)$ must pass through the point $O(0; 0)$ corresponding to the parameter $\varphi = 0$, it follows that $C_2 = r$.

We write out the brachistochrone equation in parametric form for the rolling cylinder with the above constants taken into account:

$$x(\varphi) = \frac{C_1}{2}(\varphi - \sin\varphi) - r \cos \frac{\varphi}{2} + r, \tag{3.6}$$

$$z(\varphi) = \frac{C_1}{2}(1 - \cos\varphi) + r \sin \frac{\varphi}{2}. \tag{3.7}$$

We determine C_1 from the condition that the curve passes through the point $L(x_L; z(x_L))$, i.e., the following relations must be satisfied for a certain value of the parameter $\varphi = \varphi_L$:

$$x_L = \frac{C_1}{2}(\varphi_L - \sin\varphi_L) - r \cos \frac{\varphi_L}{2} + r, \tag{3.8}$$

$$z_L = \frac{C_1}{2}(1 - \cos\varphi_L) + r \sin \frac{\varphi_L}{2}. \tag{3.9}$$

From the nonlinear system of Eqs. (3.8), (3.9) we obtain the parameter value φ_L that corresponds to the point $L(x_L; z(x_L))$ and the constant C_1 which is of dimension of length.

Using the obtained Eqs. (3.6), (3.7) for the curve $z = z(x)$, we determine the trajectory of motion of the cylinder center of mass. We write the vector relation between the vectors $\mathbf{OK}$, $\mathbf{OM}$, and $r\mathbf{n} = \mathbf{KM}$, namely, $\mathbf{OM} = \mathbf{OK} + \mathbf{KM}$.

As a result, we obtain the parametric equations for the cycloid:

$$x_M(\varphi) = \frac{C_1}{2}(\varphi - \sin\varphi) + r, \quad z(\varphi) = \frac{C_1}{2}(1 - \cos\varphi). \tag{3.10}$$

Thus, the cylinder center of mass in the case of the cylinder motion along the curve (3.6), (3.7) moves along the cycloid (3.10).

4. DETERMINING THE OPTIMAL TIME

Let the parameter $\varphi = \varphi_L$ correspond to the point $K(x_L; z(x_L))$ on the curve $z = z(x)$, which is determined by the nonlinear system of Eqs. (3.8)–(3.9). We determine the value of the functional (2.6) starting from formulas (3.6)–(3.7). After several transformations, we obtain

$$T = \sqrt{\frac{3}{4g}} \int_0^{\varphi_L} \sqrt{C_1}\, d\varphi = \sqrt{\frac{3C_1}{4g}}\varphi_L. \tag{4.1}$$

As follows from formula (4.1), the optimal time T is independent of the cylinder radius. For example, the optimal time T_{OL} in the case of the cylinder rolling from point $O(0;0)$ to point $L(r+\pi; r+2)$, which is determined by the parameter $\varphi_L = \pi$ and for which $C_1 = 2$, is equal to

$$T_{OL} = \pi\sqrt{\frac{3}{2g}}. \tag{4.2}$$

5. CONDITIONS FOR REALIZATION OF THE CYLINDER NO-SLIP MOTION WITHOUT SEPARATIONS

Let us verify the general condition for the cylinder motion along a curve: the curve radius of curvature (3.6), (3.7) at each point of their tangency must be greater than the cylinder radius.

Since the radius of curvature ρ of the curve (3.6), (3.7) is determined by the formula $\rho = 2C_1 \sin(\frac{1}{2}\varphi) + r$, it follows that this condition is satisfied ($C_1 > 0$ and $\varphi \in [0; 2\pi]$).

Now we find the conditions for the implementation of the cylinder no-slip motion without separation along the curve $z = z(x)$.

Let us write out the equations of motion of the cylinder along the curve (3.6), (3.7) with the constraint reaction $\mathbf{R}$ at point K taken into account:

$$m\dot{\mathbf{V}} = mg\mathbf{j} + \mathbf{R}, \tag{5.1}$$

$$J\ddot{\psi} = -r\mathbf{n} \times \mathbf{R}, \tag{5.2}$$

where $\mathbf{j}$ is the unit vector of the axis OZ and ψ is the angle of the cylinder rotation about the axis passing through its center of mass.

From Eq. (5.1), we find the projections of the constraint reaction $\mathbf{R}$ onto the unit normal vector $\mathbf{n} = (\cos(\varphi/2); -\sin(\varphi/2))$ and the unit tangent vector $\boldsymbol{\tau} = (\sin(\varphi/2); \cos(\varphi/2))$, which we denote by R_n and R_τ:

$$\mathbf{R} \cdot \boldsymbol{\tau} = R_\tau = m\left[\left(C_1 \sin\frac{\varphi}{2} + \frac{r}{2}\right)\ddot{\varphi} + \frac{C_1}{2}\cos\frac{\varphi}{2}\dot{\varphi}^2\right] - mg\cos\frac{\varphi}{2}, \tag{5.3}$$

$$\mathbf{R} \cdot \mathbf{n} = R_n = m\left(\frac{C_1}{2}\sin\frac{\varphi}{2} + \frac{r}{4}\right)\dot{\varphi}^2 + mg\sin\frac{\varphi}{2}. \tag{5.4}$$

From the no-slip condition $ds = r\,d\psi$ (ds is an arc element of the curve (3.6)–(3.7)), we obtain the relation between the differentials $d\psi$ and $d\varphi$:

$$ds = \sqrt{1 + (z')^2}\, dx = \left(C_1 \sin\frac{\varphi}{2} + \frac{r}{2}\right) d\varphi. \tag{5.5}$$

Thus, it follows from (5.5) that

$$d\psi = \frac{1}{r}\left(C_1 \sin\frac{\varphi}{2} + \frac{r}{2}\right) d\varphi. \tag{5.6}$$

Using (5.6), we find $\ddot{\psi}$:

$$\ddot{\psi} = \frac{1}{r}\left[\frac{C_1}{2}\cos\frac{\varphi}{2}\dot{\varphi}^2 + \left(C_1 \sin\frac{\varphi}{2} + \frac{r}{2}\right)\ddot{\varphi}\right]. \tag{5.7}$$

After several transformations with (5.1) and (5.7) taken into account, Eq. (5.2) becomes

$$\ddot{\varphi} = \frac{\frac{4}{3}g - C_1\dot{\varphi}^2}{2C_1 \sin(\varphi/2) + r}\cos\frac{\varphi}{2}. \tag{5.8}$$

We substitute the right-hand side of (5.8) into (5.3) for R_τ. As a result, we obtain

$$R_\tau = -\frac{1}{3}mg\cos\frac{\varphi}{2}. \tag{5.9}$$

Let us analyze the results. From formula (5.4), we can determine the condition under which the cylinder separation from the curve is possible:

$$m\left(\frac{C_1}{2}\sin\frac{\varphi}{2} + \frac{r}{4}\right)\dot{\varphi}^2 + mg\sin\frac{\varphi}{2} = 0. \tag{5.10}$$

Starting from (5.10), we conclude that separation is possible if $\dot{\varphi} = 0$ and $\varphi = 0$ or $\dot{\varphi} = 0$ and $\varphi = 2\pi$ simultaneously, because $R_n > 0$ for all $\varphi \in (0; 2\pi)$.

The cylinder slip along the curve arises under the following condition. We assume that the coefficient of adhesion between the cylinder and the curve at point K is equal to f_{adh}. We use the relation well known in statics:

$$|R_\tau| \leqslant F_{\text{adh}}^{\max} = f_{\text{adh}} R_n. \tag{5.11}$$

Substituting the expressions (5.4) and (5.9) into (5.11), we obtain the condition under which the cylinder motion with slip is possible:

$$\frac{\frac{1}{3}g\cos\frac{\varphi}{2}}{\left(\frac{C_1}{2}\sin\frac{\varphi}{2} + \frac{r}{4}\right)\dot{\varphi}^2 + g\sin\frac{\varphi}{2}} \leqslant f_{\text{adh}}. \tag{5.12}$$

For example, if the value of the parameter φ for the point of the cylinder contact on the curve is $\varphi = \pi$, then $R_\tau = 0$, and hence the motion with slip will be observed in a neighborhood of this point.

6. CONCLUSIONS

The results obtained in the form of differential equation (2.10) and the algebraic parametric equations (3.6)–(3.7) of the curve are continuations of the investigations performed earlier by other authors [1–19] in the framework of the variational brachistochrone problem.

In the present variational problem, we considered the fastest no-slip rolling of a heavy cylinder along a cylindrical cavity with the desired directional curve $z(x)$. The differential equation (2.10) admits quadratures in the form of parametric equations (3.6)–(3.7) for the directional curve.

On the basis of the equations of the cylinder motion with constraint reactions, we obtain realizations of its no-slip motion without separation along a brachistochrone and the optimal time, i.e., the time in which the cylinder rolls form one given point on the plane to another given point. We also show that, in the case of such rolling, the cylinder center of mass moves along a cycloid.

In the special case where the cylinder radius r is zero ($r = 0$), the obtained equation (2.10) naturally transforms into the well-known cycloid equation.

The results of this study are of both theoretical and practical interest for researchers in the field of theoretical mechanics and applied mathematics.

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