Where we are:

Chap 2  Configuration Space
Chap 3  Rigid-Body Motions
  3.2   Rotations and Angular Velocities
  3.3.1  Homogeneous Transformation Matrices
  3.3.2  Twists
Chap 4  Forward Kinematics
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Chap 9  Trajectory Generation
Chap 11 Robot Control
Chap 13 Wheeled Mobile Robots
Important concepts, symbols, and equations

• The special Euclidean group \( SE(3) \) is a matrix Lie group also known as the group of rigid-body motions or homogeneous transformation matrices in \( \mathbb{R}^3 \).

\[
T = \begin{bmatrix}
R & p \\
0 & 1
\end{bmatrix} = \begin{bmatrix}
r_{11} & r_{12} & r_{13} & p_1 \\
r_{21} & r_{22} & r_{23} & p_2 \\
r_{31} & r_{32} & r_{33} & p_3 \\
0 & 0 & 0 & 1
\end{bmatrix} \in SE(3) \quad R \in SO(3) \text{ and } p \in \mathbb{R}^3
\]

• The inverse of \( T \in SE(3) \) is

\[
T^{-1} = \begin{bmatrix}
R & p \\
0 & 1
\end{bmatrix}^{-1} = \begin{bmatrix}
R^T & -R^T p \\
0 & 1
\end{bmatrix}
\]
Important concepts, symbols, and equations (cont.)

• Three uses of HT matrices:

1. Represent a configuration. $T_{ab}$ represents frame $\{b\}$ relative to $\{a\}$.
2. Change the reference frame of a vector or frame.

\[
T_{ab}T_{bc} = T_{ac} \\
T_{ab}v_b = T_{a\phi}v_{\phi} = v_a
\]

3. Displace a vector or frame. $T = (R, p) = \text{Trans}(p) \text{ Rot}(\hat{\omega}, \theta)$

\[
\text{Trans}(p) = \begin{bmatrix}
1 & 0 & 0 & p_x \\
0 & 1 & 0 & p_y \\
0 & 0 & 1 & p_z \\
0 & 0 & 0 & 1
\end{bmatrix} \\
\text{Rot}(\hat{\omega}, \theta) = \begin{bmatrix}
R & 0 \\
0 & 1
\end{bmatrix}
\]

$v$ should be written in homogeneous coordinates, $v = [v_1 \ v_2 \ v_3 \ 1]^T$. 

Modern Robotics, Lynch and Park, Cambridge University Press
Important concepts, symbols, and equations (cont.)

Space-frame transformation:

\[ T_{sb''} = T_{sb} = \begin{vmatrix} \text{Trans}(p) & \text{Rot}(\hat{\omega}, \theta) \end{vmatrix} T_{sb} \]

2. translate \( \{b'\} \) by \( p \) in \( \{s\} \) to get \( \{b''\} \)
1. rotate \( \{b\} \) by \( \theta \) about \( \hat{\omega} \) in \( \{s\} \) to get \( \{b'\} \)
   (moves \( \{b\} \) origin)

Body-frame transformation:

\[ T_{sb''} = T_{sb} T = \begin{vmatrix} \text{Trans}(p) & \text{Rot}(\hat{\omega}, \theta) \end{vmatrix} T_{sb} \]

1. translate \( \{b\} \) by \( p \) in \( \{b\} \) to get \( \{b'\} \)
2. rotate \( \{b'\} \) by \( \theta \) about \( \hat{\omega} \) in \( \{b'\} \) to get \( \{b''\} \)
Any rigid-body velocity can be represented as a **screw axis** (a direction $\hat{s} \in S^2$, a point $q \in \mathbb{R}^3$ on the screw, and the **pitch** (linear speed/angular speed) of the screw $h$), plus the speed along the screw $\dot{\theta}$.

If $h$ is infinite, $\dot{\theta}$ is the linear speed. Otherwise, it is the angular speed.

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Important concepts, symbols, and equations (cont.)

The twist $\mathcal{V}_a = (\omega_a, v_a) \in \mathbb{R}^3$ is the angular velocity expressed in $\{a\}$ and the linear velocity of the origin of $\{a\}$ expressed in $\{a\}$.
Important concepts, symbols, and equations (cont.)

- To transform a twist from one frame to another,

\[
\mathcal{V}_a = \mathcal{T}_{ab} \mathcal{V}_b
\]

\[
\mathcal{T}_{ab} \in \mathbb{R}^{4\times4}, \quad \mathcal{V}_b \in \mathbb{R}^{6\times1}
\]

\[
\mathcal{V}_a = [\text{Ad}_{\mathcal{T}_{ab}}] \mathcal{V}_b, \text{ where the adjoint representation of } T = (R, p) \text{ is}
\]

\[
[\text{Ad}_T] = \begin{bmatrix}
R & 0 \\
[p]R & R
\end{bmatrix} \in \mathbb{R}^{6\times6}
\]
Important concepts, symbols, and equations (cont.)

Matrix representation of twists:

\[
T_{sb}^{-1} \dot{T}_{sb} = [V_b] = \begin{bmatrix} \omega_b & v_b \\ 0 & 0 \end{bmatrix} \in se(3)
\]

\[
\dot{T}_{sb} T_{sb}^{-1} = [V_s] = \begin{bmatrix} \omega_s & v_s \\ 0 & 0 \end{bmatrix} \in se(3)
\]

where \(se(3)\) is the Lie algebra of \(SE(3)\) (the set of all possible \(\dot{T}\) when \(T = I\)).
Important concepts, symbols, and equations (cont.)

Screws and twists:

- for a screw axis \( \{q, \hat{s}, h\} \) with finite \( h \),

\[
S = \begin{bmatrix}
\omega \\
v
\end{bmatrix} = \begin{bmatrix}
\hat{s} \\
-\hat{s} \times q + h\hat{s}
\end{bmatrix}
\]

- “unit” screw axis is \( S = \begin{bmatrix}
\omega \\
v
\end{bmatrix} \in \mathbb{R}^6 \),

where either (i) \( \|\omega\| = 1 \) or (ii) \( \omega = 0 \) and \( \|v\| = 1 \)

- \( \nu = S\dot{\theta} \)
Screws and twists

• for a screw axis \( \{q, \hat{s}, h\} \) with finite \( h \),

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• “unit” screw axis is \( S = \begin{bmatrix}
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where either (i) \( \|\omega\| = 1 \) or

(ii) \( \omega = 0 \) and \( \|v\| = 1 \)

• \( \mathcal{V} = S\dot{\theta} \)
For J4, what is the screw axis $S_b$? $S_s$?

For J2?
Write $T_{bc}$. 

$p_b = (3,3,0)$
A camera with frame \( \{c\} \) tracks the optical marker on a tool to get its frame \( \{t\} \) relative to \( \{c\} \). This transformation is \( T_1 \). A space frame \( \{s\} \) is attached to the floor of the room, and the camera observes its configuration relative to \( \{c\} \) as \( T_2 \). A robot arm has a mounting frame \( \{m\} \) which has been measured relative to \( \{s\} \) as \( T_3 \). The arm’s encoders and the robot’s kinematics tell us the gripper’s frame \( \{e\} \) relative to \( \{m\} \). This is represented as \( T_4 \). The gripper should be at the frame \( \{g\} \) relative to \( \{t\} \) to be able to close on the tool and pick it up. This configuration is represented as \( T_5 \).

Write \( T_{eg} \), the configuration of the grasping frame \( \{g\} \) relative to the current end-effector frame \( \{e\} \), in terms of \( T_1, T_2, T_3, T_4, \) and \( T_5 \).