Time-optimal time scaling

Chapter 9

Introduction to Robotics:

Mechanics, Planning, and Control

Frank Park and Kevin Lynch

SEOUL NATIONAL UNIVERSITY

NORTHWESTERN UNIVERSITY

Time-optimal time scaling of a path

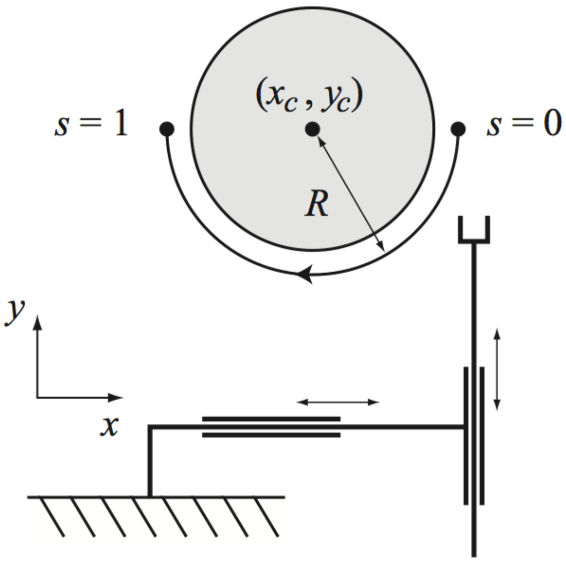


Figure 9.8: A path planner has returned a semicircular path of radius R around an obstacle in (x, y) space for a robot with two prismatic joints. The path can be represented as a function of a path parameter s as $x(s) = x_c + R\cos s\pi$ and $y(s) = y_c - R\sin s\pi$ for $s \in [0, 1]$. For a 2R robot, inverse kinematics would be used to express the path as a function of s in joint coordinates.

Dynamics constrained to a path

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + g(\theta) = \tau$$

$$C(\theta,\dot{\theta})\dot{\theta} = \dot{\theta}^T\Gamma(\theta)\dot{\theta}$$

replacing $\dot{\theta}$ by $(d\theta/ds)\dot{s}$

and $\ddot{\theta}$ by $(d\theta/ds)\ddot{s} + (d^2\theta/ds^2)\dot{s}^2$

$$\underbrace{\left(M(\theta(s))\frac{d\theta}{ds}\right)}_{m(s)}\ddot{s} + \underbrace{\left(M(\theta(s))\frac{d^2\theta}{ds^2} + \left(\frac{d\theta}{ds}\right)^T\Gamma(\theta(s))\frac{d\theta}{ds}\right)}_{c(s)}\dot{s}^2 + \underbrace{g(\theta(s))}_{g(s)} = \tau$$

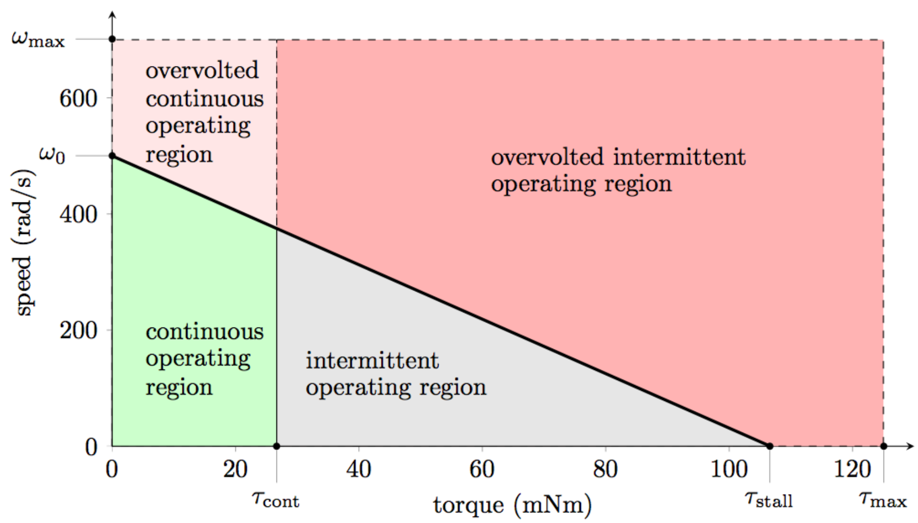
$$m(s)\ddot{s} + c(s)\dot{s}^2 + g(s) = \tau$$

SEOUL NATIONAL UNIVERSITY

NORTHWESTERN UNIVERSITY

Actuator torque/force limits

$$\tau_i^{\min}(\theta, \dot{\theta}) \leq \tau_i \leq \tau_i^{\max}(\theta, \dot{\theta})$$



$$\tau_i^{\min}(s, \dot{s}) \leq m_i(s)\ddot{s} + c_i(s)\dot{s}^2 + g_i(s) \leq \tau_i^{\max}(s, \dot{s})$$

SEOUL NATIONAL UNIVERSITY

NORTHWESTERN UNIVERSITY

Acceleration limits

$$\tau_i^{\min}(s,\dot{s}) \le m_i(s)\ddot{s} + c_i(s)\dot{s}^2 + g_i(s) \le \tau_i^{\max}(s,\dot{s})$$

$$\text{if } m_i(s) > 0: \quad L_i(s) = \frac{\tau_i^{\min}(s,\dot{s}) - c(s)\dot{s}^2 - g(s)}{m_i(s)}$$

$$U_i(s) = \frac{\tau_i^{\max}(s,\dot{s}) - c(s)\dot{s}^2 - g(s)}{m_i(s)}$$

$$\text{if } m_i(s) < 0: \quad L_i(s) = \frac{\tau_i^{\max}(s,\dot{s}) - c(s)\dot{s}^2 - g(s)}{m_i(s)}$$

$$U_i(s) = \frac{\tau_i^{\min}(s,\dot{s}) - c(s)\dot{s}^2 - g(s)}{m_i(s)}$$

if $m_i(s) = 0$: *zero-inertia point*, discussed in Section 9.4.3

The problem statement

$$L(s,\dot{s}) = \max_i L_i(s,\dot{s}) \qquad \text{and} \qquad U(s,\dot{s}) = \min_i U_i(s,\dot{s})$$

$$L(s,\dot{s}) \le \ddot{s} \le U(s,\dot{s})$$

Given a path $\theta(s), s \in [0,1]$, *an initial state* $(s_0, \dot{s}_0) = (0,0)$, *and a final state* $(s_f, \dot{s}_f) = (1,0)$, *find a monotonically increasing twice-differentiable time scaling* $s : [0,T] \to [0,1]$ *that*

(i) *satisfies* $s(0) = \dot{s}(0) = \dot{s}(T) = 0$ *and* $s(T) = 1$, *and*

(ii) *minimizes the total travel time* T *along the path while respecting the actuator constraints* (9.37).

SEOUL NATIONAL UNIVERSITY

NORTHWESTERN UNIVERSITY

The $(s, \dot{s})$ phase plane

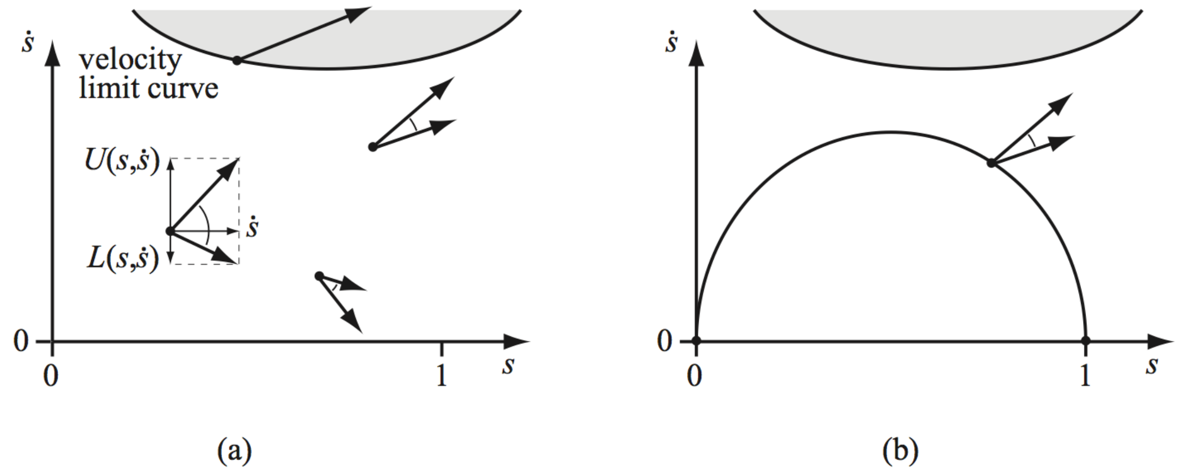
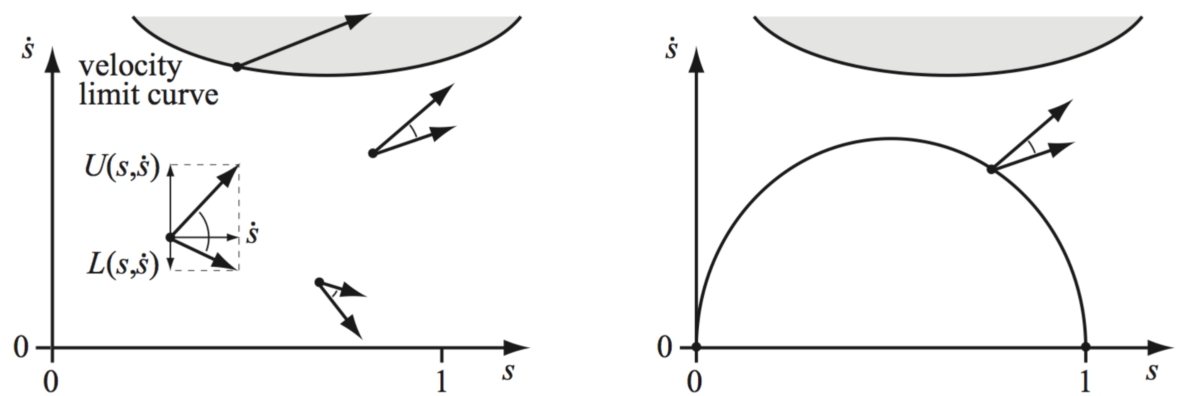


Figure 9.10: (a) Acceleration-limited motion cones at four different states. The upper ray of the cone is the sum of $U(s, \dot{s})$ plotted in the vertical direction (change in velocity) and $\dot{s}$ plotted in the horizontal direction (change in position). The lower ray of the cone is constructed from $L(s, \dot{s})$ and $\dot{s}$. Points in grey, bounded by the velocity limit curve, have $L(s, \dot{s}) \geq U(s, \dot{s})$—the state is inadmissible. On the velocity limit curve, the cone is reduced to a single tangent vector. (b) The proposed time scaling is infeasible because the tangent to the curve is outside the motion cone at the state indicated.

Maximize "speed" $\dot{s}$ at all s while satisfying acceleration and boundary constraints

$$T = \int_0^T 1\ dt$$

$$T = \int_0^t 1\ dt = \int_0^T \frac{ds}{ds}\ dt = \int_0^T \frac{dt}{ds}\ ds = \int_0^1 \dot{s}^{-1}(s)\ ds$$



(a) (b)

SEOUL NATIONAL UNIVERSITY

NORTHWESTERN UNIVERSITY

Time-optimal "bang-bang" time scaling

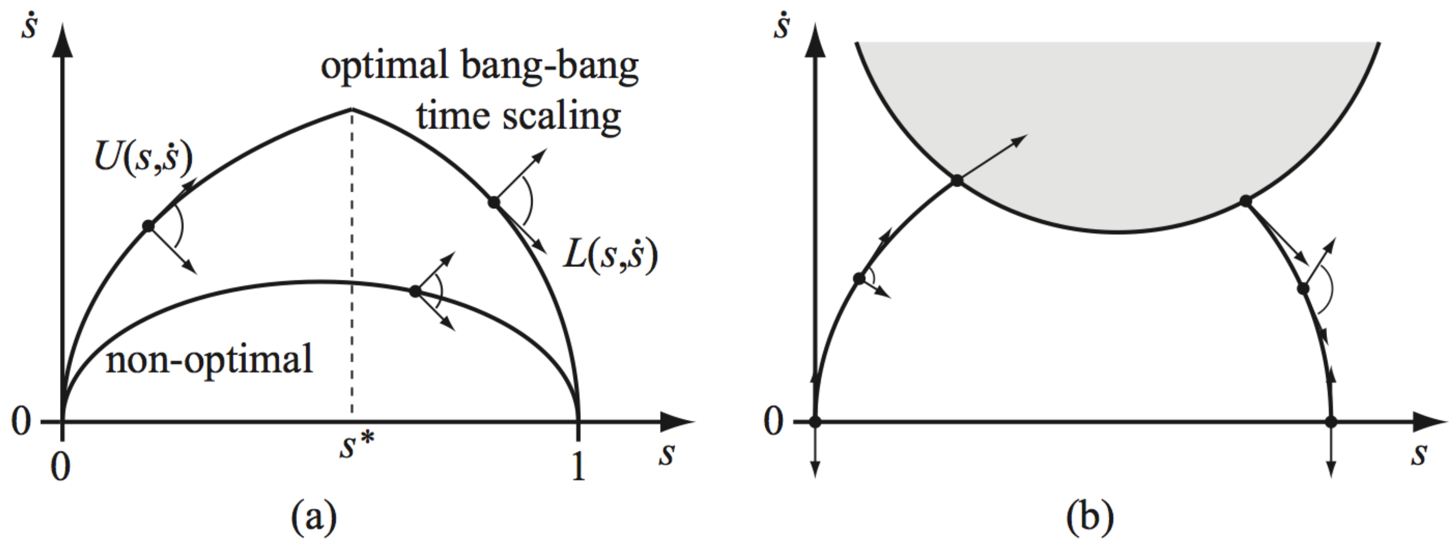


Figure 9.11: (a) A time-optimal bang-bang time scaling integrates $U(s,\dot{s})$ from $(0,0)$ and switches to $L(s,\dot{s})$ at a switching point s^*. Also shown is a non-optimal time scaling with a tangent inside a motion cone. (b) Sometimes the velocity limit curve prevents a single-switch solution.

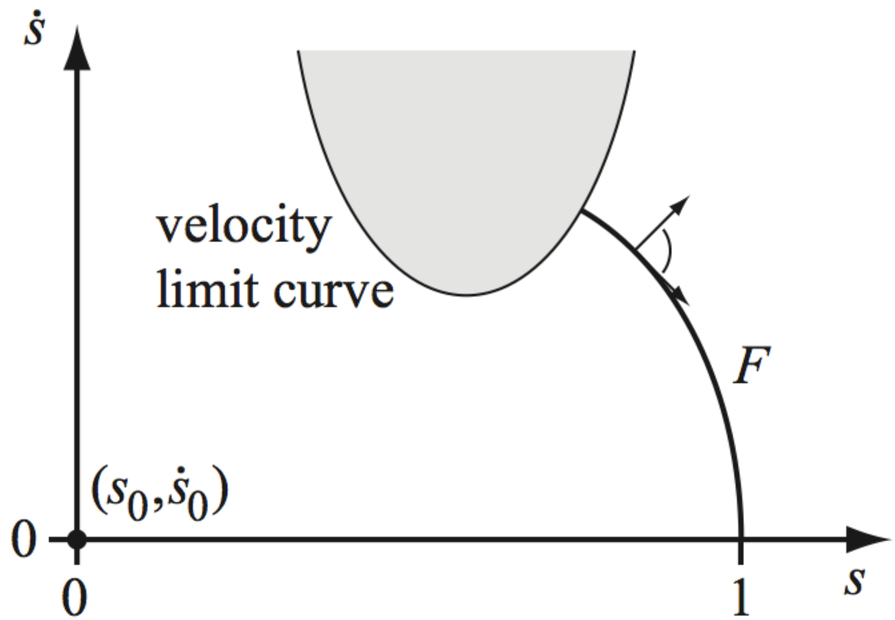
SEOUL NATIONAL UNIVERSITY

NORTHWESTERN UNIVERSITY

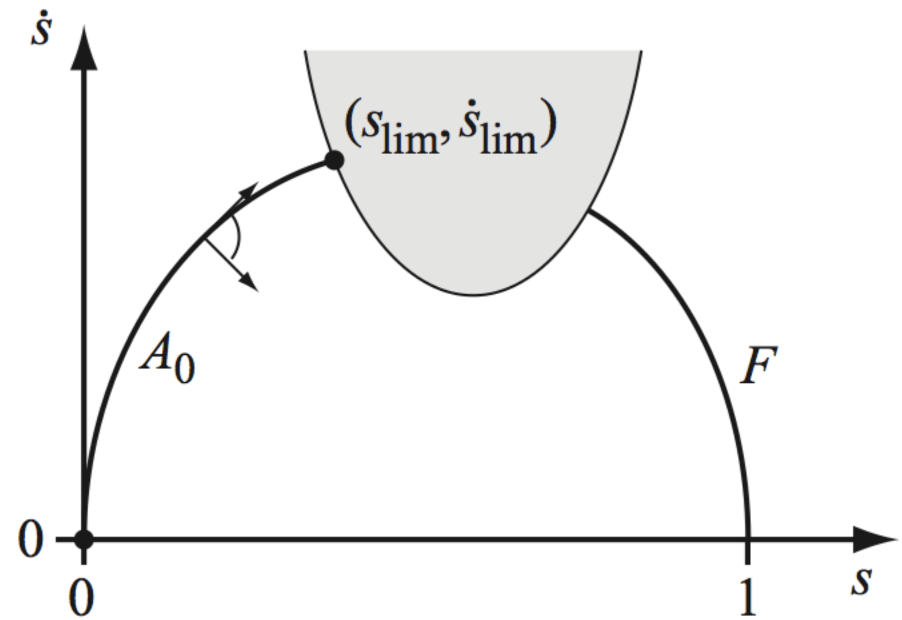
Time-scaling algorithm

1. Initialize empty list of switches S between U and L.
2. Integrate backward from end along L.
3. Integrate forward from start along U. If the curve crosses the final curve, switch U to L occurs there. Done.
4. If speed limit is exceeded, lower the speed at the penetration s until velocity limit not reached when integrating forward along U. Call the point just touching the speed limit B.
5. Integrate backward along L from B until intersecting the previous U motion segment. Switch to L occurs there.
6. Switch to U occurs at B. Set "start" to B, go to step 3.

Steps 2 and 3



Step 2: $i = 0$, $S = \{\}$

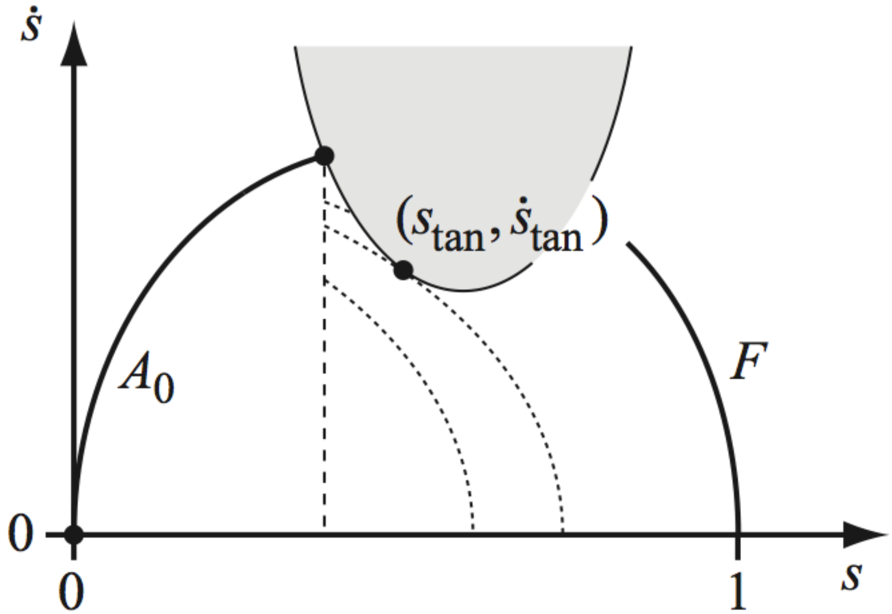


Step 3: $i = 0$, $S = \{\}$

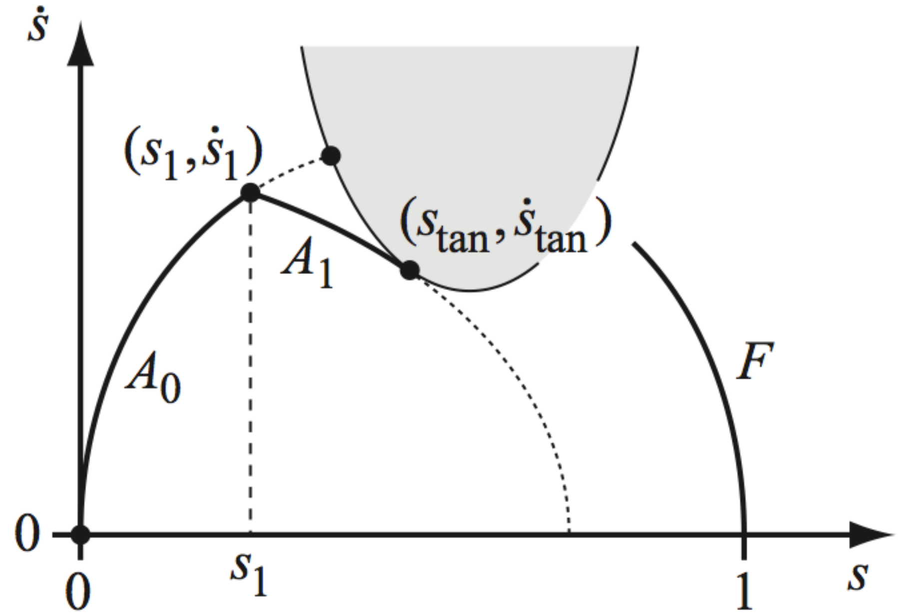
SEOUL NATIONAL UNIVERSITY

NORTHWESTERN UNIVERSITY

Steps 4 and 5



Step 4: $i = 0$, $S = \{\}$

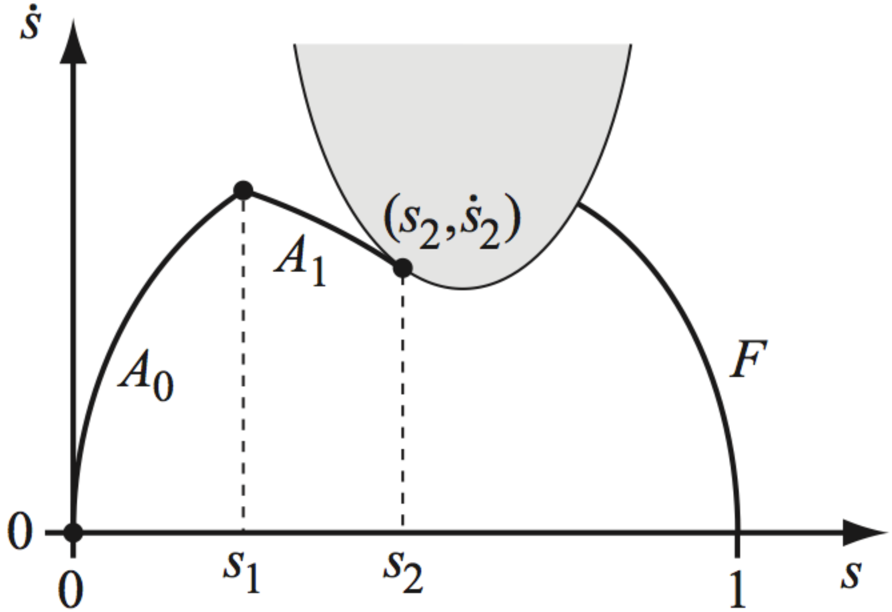


Step 5: $i = 1$, $S = \{s_1\}$

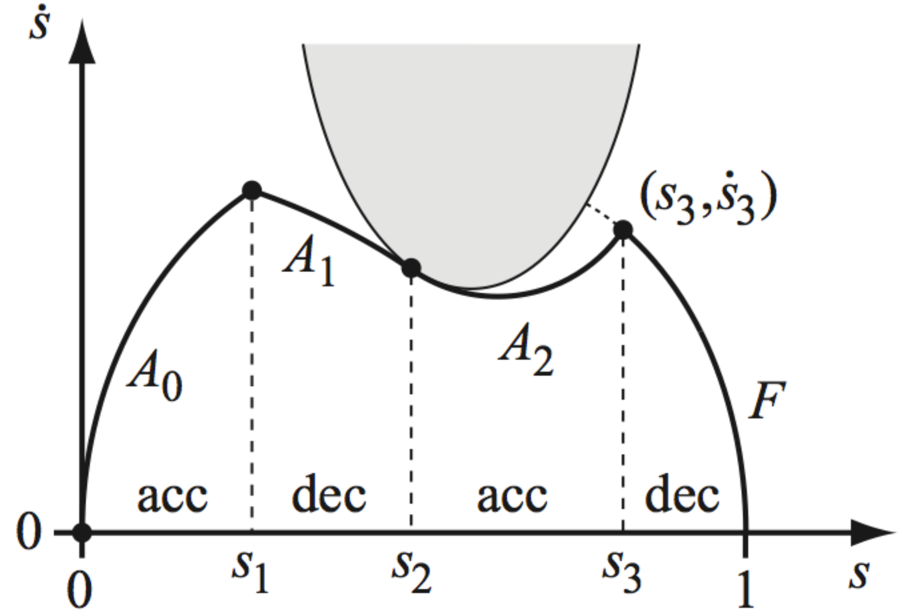
SEOUL NATIONAL UNIVERSITY

NORTHWESTERN UNIVERSITY

Step 6, and back to step 3



Step 6: $i = 2$, $S = \{s_1, s_2\}$



Step 3: $i = 3$, $S = \{s_1, s_2, s_3\}$

SEOUL NATIONAL UNIVERSITY

NORTHWESTERN UNIVERSITY

Assumptions

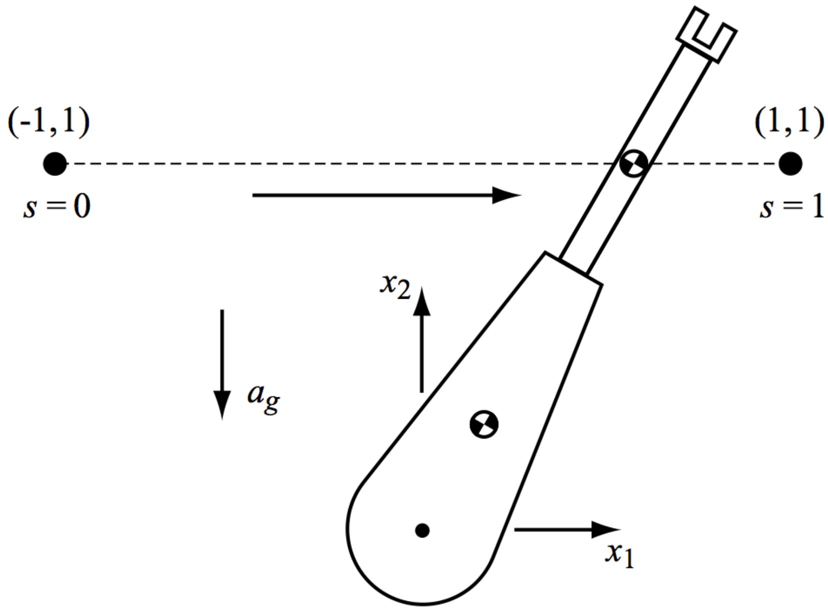
- Actuators are strong enough to hold static posture at all s
- Single speed limit for all s
- No zero-inertia points (zip) where one or more $m_i = 0$

if isolated zips, or a "singular arc" of zips, then some actuator speed constraints come directly from

$$\tau_i^{\min}(s,\dot{s}) \leq \overset{\nearrow 0}{m_i(s)}\ddot{s} + c_i(s)\dot{s}^2 + g_i(s) \leq \tau_i^{\max}(s,\dot{s})$$

and others from $L_i(s,\dot{s}) = U_i(s,\dot{s})$

Zero-inertia point



for actuator 2 at $s = 1/2$