Time-optimal time scaling

Chapter 9
Introduction to Robotics: Mechanics, Planning, and Control
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Time-optimal time scaling of a path

Figure 9.8: A path planner has returned a semicircular path of radius $R$ around an obstacle in $(x, y)$ space for a robot with two prismatic joints. The path can be represented as a function of a path parameter $s$ as $x(s) = x_c + R \cos s\pi$ and $y(s) = y_c - R \sin s\pi$ for $s \in [0, 1]$. For a 2R robot, inverse kinematics would be used to express the path as a function of $s$ in joint coordinates.
Dynamics constrained to a path

\[ M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + g(\theta) = \tau, \]

\[ C(\theta, \dot{\theta})\dot{\theta} = \dot{\theta}^T \Gamma(\theta) \dot{\theta} \]

replacing \( \dot{\theta} \) by \( (d\theta/ds)\dot{s} \)

and \( \ddot{\theta} \) by \( (d\theta/ds)\ddot{s} + (d^2\theta/ds^2)\dot{s}^2 \)

\[ \begin{align*}
    \left( M'(\theta(s)) \frac{d\theta}{ds} \right) \ddot{s} + \left( M'(\theta(s)) \frac{d^2\theta}{ds^2} + \left( \frac{d\theta}{ds} \right)^T \Gamma(\theta(s)) \frac{d\theta}{ds} \right) \dot{s}^2 + g(\theta(s)) &= \tau \\
    m(s)\ddot{s} + c(s)\dot{s}^2 + g(s) &= \tau,
\end{align*} \]
Actuator torque/force limits

\[ \tau_i^{\text{min}}(\theta, \dot{\theta}) \leq \tau_i \leq \tau_i^{\text{max}}(\theta, \dot{\theta}) \]

\[ \tau_i^{\text{min}}(s, \dot{s}) \leq m_i(s)\ddot{s} + c_i(s)\dot{s}^2 + g_i(s) \leq \tau_i^{\text{max}}(s, \dot{s}) \]
Acceleration limits

\[ \tau_i^{\min}(s, \dot{s}) \leq m_i(s)\ddot{s} + c_i(s)\dot{s}^2 + g_i(s) \leq \tau_i^{\max}(s, \dot{s}) \]

if \( m_i(s) > 0 \):
\[ L_i(s) = \frac{\tau_i^{\min}(s, \dot{s}) - c(s)\dot{s}^2 - g(s)}{m_i(s)} \]
\[ U_i(s) = \frac{\tau_i^{\max}(s, \dot{s}) - c(s)\dot{s}^2 - g(s)}{m_i(s)} \]

if \( m_i(s) < 0 \):
\[ L_i(s) = \frac{\tau_i^{\max}(s, \dot{s}) - c(s)\dot{s}^2 - g(s)}{m_i(s)} \]
\[ U_i(s) = \frac{\tau_i^{\min}(s, \dot{s}) - c(s)\dot{s}^2 - g(s)}{m_i(s)} \]

if \( m_i(s) = 0 \):  \textit{zero-inertia point}, discussed in Section 9.4.3
The problem statement

\[ L(s, \dot{s}) = \max_i L_i(s, \dot{s}) \quad \text{and} \quad U(s, \dot{s}) = \min_i U_i(s, \dot{s}) \]

\[ L(s, \dot{s}) \leq \ddot{s} \leq U(s, \dot{s}) \]

Given a path \( \theta(s), s \in [0, 1] \), an initial state \( (s_0, \dot{s}_0) = (0, 0) \), and a final state \( (s_f, \dot{s}_f) = (1, 0) \), find a monotonically increasing twice-differentiable time scaling \( s : [0, T] \to [0, 1] \) that

(i) satisfies \( s(0) = \dot{s}(0) = \ddot{s}(T) = 0 \) and \( s(T) = 1 \), and

(ii) minimizes the total travel time \( T \) along the path while respecting the actuator constraints (9.37).
The \((s, \dot{s})\) phase plane

Figure 9.10: (a) Acceleration-limited motion cones at four different states. The upper ray of the cone is the sum of \(U(s, \dot{s})\) plotted in the vertical direction (change in velocity) and \(\dot{s}\) plotted in the horizontal direction (change in position). The lower ray of the cone is constructed from \(L(s, \dot{s})\) and \(\dot{s}\). Points in grey, bounded by the velocity limit curve, have \(L(s, \dot{s}) \geq U(s, \dot{s})\)—the state is inadmissible. On the velocity limit curve, the cone is reduced to a single tangent vector. (b) The proposed time scaling is infeasible because the tangent to the curve is outside the motion cone at the state indicated.
Maximize “speed” $\dot{s}$ at all $s$ while satisfying acceleration and boundary constraints

$$T = \int_0^T 1 \, dt$$

$$T = \int_0^t 1 \, dt = \int_0^T \frac{ds}{ds} \, dt = \int_0^T \frac{dt}{ds} \, ds = \int_0^1 \dot{s}^{-1}(s) \, ds$$
Time-optimal “bang-bang” time scaling

Figure 9.11: (a) A time-optimal bang-bang time scaling integrates $U(s, \dot{s})$ from $(0,0)$ and switches to $L(s, \dot{s})$ at a switching point $s^*$. Also shown is a non-optimal time scaling with a tangent inside a motion cone. (b) Sometimes the velocity limit curve prevents a single-switch solution.
Time-scaling algorithm

1. Initialize empty list of switches $S$ between $U$ and $L$.
2. Integrate backward from end along $L$.
3. Integrate forward from start along $U$. If the curve crosses the final curve, switch $U$ to $L$ occurs there. Done.
4. If speed limit is exceeded, lower the speed at the penetration $s$ until velocity limit not reached when integrating forward along $U$. Call the point just touching the speed limit $B$.
5. Integrate backward along $L$ from $B$ until intersecting the previous $U$ motion segment. Switch to $L$ occurs there.
6. Switch to $U$ occurs at $B$. Set “start” to $B$, go to step 3.
Steps 2 and 3

Step 2: $i = 0, S = \{\}$

Step 3: $i = 0, S = \{\}$
Steps 4 and 5

Step 4: $i = 0$, $S = \{\}$

Step 5: $i = 1$, $S = \{s_1\}$
Step 6, and back to step 3

Step 6: \( i = 2, S = \{s_1, s_2\} \)

Step 3: \( i = 3, S = \{s_1, s_2, s_3\} \)
Assumptions

• Actuators are strong enough to hold static posture at all $s$
• Single speed limit for all $s$
• No zero-inertia points (zip) where one or more $m_i = 0$
  if isolated zips, or a “singular arc” of zips, then some actuator speed constraints come directly from

$$\tau_i^{\text{min}}(s, \dot{s}) \leq m_i(s)\ddot{s} + c_i(s)\dot{s}^2 + g_i(s) \leq \tau_i^{\text{max}}(s, \dot{s})$$

and others from $L_i(s, \dot{s}) = U_i(s, \dot{s})$
Zero-inertia point

for actuator 2 at $s = 1/2$