ME449 – Problem Set 4 – Solution Set

Question 1

Introduction :

Using the example a simple 2R robot, this document demonstrates the process of applying the Newton-Euler method to determine the inverse dynamics. The example is meant to deliver a core understanding of the method. It can then be easliy generalized and applied to more comlex systems.

Preliminaries :

Observe the drawing. Find the transformation matrix $M_i \in SE(3)$ for each link. M_i is the transformation from the base frame {0} to the frame {i}, which is attached to the center of mass of the i-th link, when the robot is in its home configuration.

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & L1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 0 & L1 + L2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$M_{12} \in SE(3)$ is the transformation matrix from the frame {1} (attached to center of mass of link 1) to the frame {2} (attached to the center of mass of link 2), when the arm is in its home configuration. Find M_{12} by observing the drawing or by using the equation $M_{12} = M_1^{-1} M_2$

$$M_{12} = \begin{pmatrix} 1 & 0 & 0 & L2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A_i is the twist-vector for joint i expressed in the frame {i} when the arm is in its home configuration ($\theta_i = 0$). For a simple RR arm it can be obtained by observing the spatial velocity of frame {i} when rotating about joint i from the home configuration. Alternatively one may use the equation $A_i = Ad_{M_i^{-1}} S_i$.

$$A_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ L1 \\ 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ L2 \\ 0 \end{pmatrix}$$

From observing the drawing, obtain the screw-axis S_i for each joint, expressed in the space-frame. We've done that plenty of times in class.

$$S_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad S_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L1 \\ 0 \end{pmatrix}$$

Define the gravity vector $a_g = \begin{pmatrix} 0 \\ g \\ 0 \end{pmatrix}$ with $g<0$.

Define the spatial inertia matrix G_i for each link i, expressed in the frame {i}. In the case of the RR robot we assume that the mass is concentrated as a point mass at the end of each link; it happens to be located at the origin of the frame {i}. Therefore, relative to the frame {i}, the mass of the link {i} has no rotational inertia.

$$G_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m1 & 0 & 0 \\ 0 & 0 & 0 & 0 & m1 & 0 \\ 0 & 0 & 0 & 0 & 0 & m1 \end{pmatrix} \quad G_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m2 & 0 & 0 \\ 0 & 0 & 0 & 0 & m2 & 0 \\ 0 & 0 & 0 & 0 & 0 & m2 \end{pmatrix}$$

The base is fixed to the ground. It therefore has no velocity. It is however subject to gravity. The gravity vector a_g needs to be incorporated in $\dot{V}_0$

$$V_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \dot{V}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ g \\ 0 \end{pmatrix}$$

Forward Iteration :

During the forward iteration of Newton-Euler inverse dynamics, we obtain the states and accelerations of the frames attached to each link. Because the velocity and acceleration of each link is influenced by those of its predecessors, we start our calculations at the base and incrementally move out-board until the states and accelerations for each link have been obtained. As a convention we will express velocities V_i and accelaratoins $\dot{V}_i$ for each link i in the frame {i}, which is attached to the center of mass of the respective link.

Link 1 states and acceleration:

We now calculate the transformation T_{01} from link {1}'s predecessor (frame{0})to itself (frame{1}).

The equation $T_{01}=M_1 e^{[A_1]\theta_1}$ takes M_1 (the transformation from the base frame {0} to the frame {1} when the robot is in its home configuration $\theta_1=\theta_2=0$) as a reference point, and incorporates twists (exponential coordinates $A_1\theta_1$) about joint 1 to find the transformation from frame {0} to frame {1} for any given θ_1

$$T_{01}=\begin{pmatrix} \text{Cos}[\theta 1] & -\text{Sin}[\theta 1] & 0 & \text{L1 Cos}[\theta 1] \\ \text{Sin}[\theta 1] & \text{Cos}[\theta 1] & 0 & \text{L1 Sin}[\theta 1] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Calculate the absolute velocity V_1 of the frame {1} expressed in frame {1}.

Detail: V_1 is composed of two terms: $V_1=Ad_{T_{10}}V_0+A_1\dot{\theta}_1$.

First term: If joint 1 had a constant angle θ_1, then the base, together with the first link, could be regarded as a single rigid body. Using the Adjoint of a transformation matrix T_{10} (between two frames {1} and {0},that are assumed to be fixed to a rigid body) a spacial velocity of one point (ie. frame {1}) can be expressed in terms of the spacial velocity of another point (ie. frame {0}). The first term considers the portion of V_1, as a result of being attached to a previous dynamic body. In this particular case, the body (base) is stationary, V_0 is 0 and therefore the first term of the equation is also 0.

Second Term: The joint angle θ_1 of joint 1 is generally not constant and the joint-angle velocity $\dot{\theta}_1$ is not 0. The second term of the equation for V_1 incorporates the additional velocity of the frame {1} caused by rotating about joint 1.

$$V_1=\begin{pmatrix} 0 \\ 0 \\ \theta 1'[t] \\ 0 \\ \text{L1}\,\theta 1'[t] \\ 0 \end{pmatrix}$$

Calculate the absolute acceleration $\dot{V}1$ of the frame {1} expressed in frame {1}.

Detail: $\dot{V}_1=Ad_{T10}\dot{V}_0+[V_1,A_1]\dot{\theta}_1+A_1\ddot{\theta}_1$, where
$[V_1,A_1]$ indicates the Lie-Bracket operation of V_1 and A_1.
First term: Considers the acceleration of the previous rigid body, ie. the acceleration of the base frame {0}.
Second term: Considers the corriolis and centripital accelerations.
Third term: Considers accelerations of frame{1}, due to joint-angle accelerations $\ddot{\theta}_1$.

$$\dot{V}_1 = \begin{pmatrix} 0 \\ 0 \\ \theta1''[t] \\ g\,\mathrm{Sin}[\theta1] \\ g\,\mathrm{Cos}[\theta1] + L1\,\theta1''[t] \\ 0 \end{pmatrix}$$

Link 2 states and acceleration:

We now calculate the transformation T_{12} from link {2}'s predecessor (frame{1})to itself (frame{2}).

The equation $T_{12}=M_{12}e^{[A_2]\,\theta_2}$ takes M_{12} (the transformation from frame {1} to frame {2} when the robot is in its home configuration $\theta_1=\theta_2=0$) as a reference point, and incorporates twists (exponential coordinates $A_2\theta_2$) about joint 2 to find the transformation from frame {1} to frame {2} for any given θ_2.

$$T_{12} = \begin{pmatrix} \mathrm{Cos}[\theta2] & -\mathrm{Sin}[\theta2] & 0 & L2\,\mathrm{Cos}[\theta2] \\ \mathrm{Sin}[\theta2] & \mathrm{Cos}[\theta2] & 0 & L2\,\mathrm{Sin}[\theta2] \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Calculate the absolute velocity V_2 of the frame {2} expressed in frame {2}.

Detail: V_2 is composed of two terms: $V_2=\mathrm{Ad}_{T_{21}}V_1+A_2\dot{\theta}_2$

First term: If joint 2 had a constant angle θ_2, then link 1, together with link 2, could be regarded as a single rigid body. Using the Adjoint of a transformation matrix T_{21} (between two frames {2} and {1},that are assumed to be fixed to a rigid body) a spacial velocity of one point (ie. frame {2}) can be expressed in terms of the spacial velocity of another point (ie. frame {1}). The first term considers the portion of V_2, as a result of being attached to a previous dynamic body.

Second Term: The joint angle θ_2 of joint 2 is generally not constant and the joint-angle velocity $\dot{\theta}_2$ is not 0. The second term of the equation for V_2 incorporates the additional velocity of the frame {2} caused by rotating about joint 2.

$$V_2 = \begin{pmatrix} 0 \\ 0 \\ \theta1'[t] + \theta2'[t] \\ L1\,\mathrm{Sin}[\theta2]\,\theta1'[t] \\ (L2 + L1\,\mathrm{Cos}[\theta2])\,\theta1'[t] + L2\,\theta2'[t] \\ 0 \end{pmatrix}$$

Calculate the absolute acceleration $\dot{V}_2$ of the frame {2} expressed in frame {2}:

Detail: $\dot{V}_2=\mathrm{Ad}_{T21}\dot{V}_1+[V_2,A_2]\dot{\theta}_2+A_2\ddot{\theta}_2$, where
$[V_2,A_2]$ indicates the Lie-Bracket operation of V_2 and A_2.
First term: Considers the acceleration of the previous rigid body, ie. the acceleration of the frame {1}.
Second term: Considers the corriolis and centripital accelerations.
Third term: Considers accelerations of frame{2} due to joint angle accelerations $\ddot{\theta}_2$.

$$\dot{V}_2 = \begin{pmatrix} 0 \\ 0 \\ \theta1''[t] + \theta2''[t] \\ g\,Sin[\theta1+\theta2] + L1\,Cos[\theta2]\,\theta1'[t]\,\theta2'[t] + L1\,Sin[\theta2]\,\theta1''[t] \\ g\,Cos[\theta1+\theta2] - L1\,Sin[\theta2]\,\theta1'[t]\,\theta2'[t] + (L2 + L1\,Cos[\theta2])\,\theta1''[t] + L2\,\theta2''[t] \\ 0 \end{pmatrix}$$

Backward Iteration :

During the forward iteration, we had to bear in mind that the velocity and acceleration of link i is dependent on those of its predecessor link i-1. Similarly, the forces acting on a link i are dependent on the forces acting on its outboard follower (i+1): The wrench F_i that must be applied to link i is the sum of the wrench F_{i+1} that must be provided to link i+1 (but expressed in frame {i}) plus the extra wrench from the rigid body dynamics of link i. In other words, the inboard links needs to support the outboard links. For that reason, we run a backward iteration starting with the most outboard link and incrementally approach the base link. Ultimately, we are solving for the torques acting on each joint, to obtain the control torques that need to be provided to the joint motors.

Link 2 forces and torques:

Find the transform from frame{2} to frame {3}. The frame n+1 (in this case 3) is the frame attached to the end-effector. For the 2R arm we assumed that the frame {3} coincides with frame {2} (both the center of mass of link 2, as well as the end-effector are located at the tip of link 2). We therefore know, that T_{23} is the identity transformation in SE(3)

$$T_{23} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

F_3 is the wrench that the end-effector applies to the environment expressed in the frame {3}. It is given as 0.

$$F_{tip} = F_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Calculate F_2 using the eqtation $F_2 = Ad_{T32} F_{tip} + G_2 \dot{V}_2 - ad_{V_2}{}^{T}(G_2 V_2)$.
Details:
First term: The wrench that must be provided by the end-effector (expressed in frame {2}.)
Second and third term: The wrench that results from the dynamics of link 2.

$$F_2=\begin{pmatrix} 0 \\ 0 \\ 0 \\ m2\left(g\,Sin[\theta 1+\theta 2]-L1\,Cos[\theta 2]\,\theta 1'[t]^2-L2\,(\theta 1'[t]+\theta 2'[t])^2+L1\,Sin[\theta 2]\,\theta 1''[t]\right) \\ m2\left(g\,Cos[\theta 1+\theta 2]+L1\,Sin[\theta 2]\,\theta 1'[t]^2+(L2+L1\,Cos[\theta 2])\,\theta 1''[t]+L2\,\theta 2''[t]\right) \\ 0 \end{pmatrix}$$

To obtain the torque acting on joint 2, the wrench F_2 is multiplied with the twist-vector for joint 2 expressed in the frame {2}, namely A_2

$$\tau 2=F_2{}^{T}A_2=\left(L2\,m2\left(g\,Cos[\theta 1+\theta 2]+L1\,Sin[\theta 2]\,\theta 1'[t]^2+(L2+L1\,Cos[\theta 2])\,\theta 1''[t]+L2\,\theta 2''[t]\right)\right)$$

Link 1 forces and torques are obtained analogeous to link 2

$$F_1=\begin{pmatrix} 0 \\ 0 \\ L2\,m2\left(g\,Cos[\theta 1+\theta 2]+L1\,Sin[\theta 2]\,\theta 1'[t]^2+(L2+L1\,Cos[\theta 2])\,\theta 1''[t]+L2\,\theta 2''[t]\right) \\ (m1+m2)\left(g\,Sin[\theta 1]-L1\,\theta 1'[t]^2\right)-L2\,m2\,Cos[\theta 2]\,(\theta 1'[t]+\theta 2'[t])^2-L2\,m2\,Sin[\theta 2]\,(\theta 1''[t]+\theta 2''[\\ -L2\,m2\,Sin[\theta 2]\,(\theta 1'[t]+\theta 2'[t])^2+(m1+m2)\,(g\,Cos[\theta 1]+L1\,\theta 1''[t])+L2\,m2\,Cos[\theta 2]\,(\theta 1''[t]+\theta 2''[\\ 0 \end{pmatrix}$$

$$\tau_1=\left(g\,L1\,(m1+m2)\,Cos[\theta 1]+g\,L2\,m2\,Cos[\theta 1+\theta 2]-2\,L1\,L2\,m2\,Sin[\theta 2]\,\theta 1'[t]\,\theta 2'[t]-L1\,L2\,m2\,Sin[\theta 2]\,\theta 2'\right.$$

When examining the joint torques τ_1 and τ_2, we get the same answer as with the lagrangian approach. The Newton-Euler inverse dynamics have been succesfully demonstrated for the 2R arm!