Chapter 8

Practice Exercises on Dynamics of Open Chains

8.1 Practice Exercises

Practice exercise 8.1 Figure 8.1 illustrates an RP robot moving in a vertical plane. The mass of link 1 is $\mathfrak{m}_1$ and the center of mass is a distance L_1 from joint 1. The scalar inertia of link 1 about an axis through the center of mass and out of the plane is $\mathcal{I}_1$. The mass of link 2 is $\mathfrak{m}_2$, the center of mass is a distance θ_2 from joint 1, and the scalar inertia of link 2 about its center of mass is $\mathcal{I}_2$. Gravity g acts downward on the page.

(a) Let the location of the center of mass of link i be (x_i, y_i). Find (x_i, y_i) for $i = 1, 2$, and their time derivatives, in terms of θ and $\dot{\theta}$.

(b) Write the potential energy of each of the two links, $\mathcal{P}_1$ and $\mathcal{P}_2$, using the

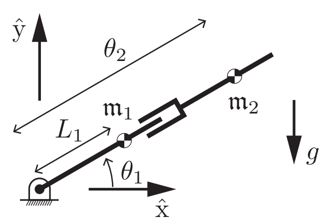

Figure 8.1: An RP robot operating in a vertical plane.

joint variables θ.

(c) Write the kinetic energy of each of the two links, $\mathcal{K}_1$ and $\mathcal{K}_2$. (Recall that the kinetic energy of a rigid body moving in the plane is $\mathcal{K} = (1/2)\mathfrak{m}v^2 + (1/2)\mathcal{I}\omega^2$, where $\mathfrak{m}$ is the mass, v is the scalar linear velocity at the center of mass, ω is the scalar angular velocity, and $\mathcal{I}$ is the scalar inertia of the rigid body about its center of mass.)

(d) What is the Lagrangian in terms of $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{P}_1$, and $\mathcal{P}_2$?

(e) One of the terms in the Lagrangian can be expressed as

$$\frac{1}{2}\mathfrak{m}_2\theta_2^2\dot{\theta}_1^2.$$

If this were the complete Lagrangian, what would the equations of motion be? Derive these by hand (no symbolic math software assistance). Indicate which of the terms in your equations are a function of $\ddot{\theta}$, which are Coriolis terms, which are centripetal terms, and which are gravity terms, if any.

(f) Now derive the equations of motion (either by hand or using symbolic math software for assistance) for the full Lagrangian and put them in the form

$$\tau = M(\theta)\ddot{\theta} + c(\theta, \dot{\theta}) + g(\theta).$$

Identify which of the terms in $c(\theta, \dot{\theta})$ are Coriolis and which are centripetal. Explain as if to someone who is unfamiliar with dynamics why these terms contribute to the joint forces and torques.

(g) Consider the configuration-dependent mass matrix $M(\theta)$ from your previous answer. When the robot is at rest (and ignoring gravity), the mass matrix can be visualized as the ellipse of joint forces/torques that are required to generate the unit circle of joint accelerations in $\ddot{\theta}$ space. As θ_2 increases, how does this ellipse change? Describe it in text and provide a drawing.

(h) Now visualize the configuration-dependent end-effector mass matrix $\Lambda(\theta)$, where the "end-effector" is considered to be at the point (x_2, y_2), the location of the center of mass of the second link. For a unit circle of accelerations $(\ddot{x}_2, \ddot{y}_2)$, consider the ellipse of linear forces that are required to be applied at the end-effector to realize these accelerations. How does the orientation of this ellipse change as θ_1 changes? How does the shape change as θ_2 increases from zero to infinity when $\theta_1 = 0$? Provide a drawing for the case $\theta_1 = 0$. If you have access to symbolic computation software (e.g., Mathematica), you can use the Jacobian $J(\theta)$ satisfying

$$\begin{bmatrix} \dot{x}_2 \\ \dot{y}_2 \end{bmatrix} = J(\theta)\dot{\theta}$$

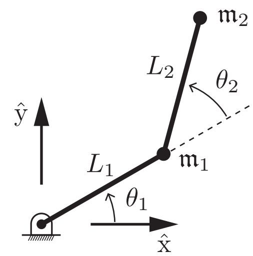

Figure 8.2: A 2R robot with all mass concentrated at the ends of the links.

to calculate $\Lambda(\theta) = J^{-\mathrm{T}}(\theta)M(\theta)J^{-1}(\theta)$ for the case $\theta_1 = 0$. If you do not have access to symbolic computation software, you can plug in numerical values for $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathfrak{m}_1$, $\mathfrak{m}_2$, and L_1 (make them all equal to 1, for example) to say something about how Λ changes (and therefore how the ellipse changes) as θ_2 goes from zero to infinity while $\theta_1 = 0$.

Practice exercise 8.2 The mass matrix of the 2R robot of Figure 8.2 is

$$M(\theta) = \begin{bmatrix} \mathfrak{m}_1 L_1^2 + \mathfrak{m}_2(L_1^2 + 2L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) \\ \mathfrak{m}_2(L_1L_2\cos\theta_2 + L_2^2) & \mathfrak{m}_2 L_2^2 \end{bmatrix},$$

where each link is modeled as a point mass at the end of the link. Explain in text and/or figures why each of the entries makes sense, for example using the joint accelerations $\ddot{\theta} = (1,0)$ and $(0,1)$.

Practice exercise 8.3 The equations of motion for a particular 2R robot arm can be written $M(\theta)\ddot{\theta} + c(\theta,\dot{\theta}) + g(\theta) = \tau$. The Lagrangian $\mathcal{L}(\theta,\dot{\theta})$ for the robot can be written in components as

$$\mathcal{L}(\theta,\dot{\theta}) = \mathcal{L}^1(\theta,\dot{\theta}) + \mathcal{L}^2(\theta,\dot{\theta}) + \mathcal{L}^3(\theta,\dot{\theta}) + \dots$$

One of these components is $\mathcal{L}^1 = \mathfrak{m}\dot{\theta}_1\dot{\theta}_2\cos\theta_2$.

(a) Find the joint torques τ_1 and τ_2 corresponding to the component $\mathcal{L}^1$.

(b) Write the 2×2 mass matrix $M^1(\theta)$, the velocity-product vector $c^1(\theta,\dot{\theta})$, and the gravity vector $g^1(\theta)$ corresponding to $\mathcal{L}^1$. (Note that $M = M^1 + M^2 + M^3 + \dots$, $c = c^1 + c^2 + c^3 + \dots$, and $g = g^1 + g^2 + g^3 + \dots$)

Practice exercise 8.4 For a given configuration θ of a two-joint robot, the mass matrix is

$$M(\theta) = \begin{bmatrix} 3 & a \\ b & 2 \end{bmatrix},$$