

①

In an n -dimensional space, a rigid body has $n + (n-1) + (n-2) + \dots + 1$ dof

In that equation, I have n terms. I will express the same expression as a sum of the outermost components (and moving towards the middle). That is:

$$\underbrace{n + (n-1) + (n-2) + \dots + 1}_{n \text{ terms}} = \underbrace{(n+1) + [(n-1)+2] + \dots}_{n/2 \text{ terms}}$$

$$= \underbrace{(n+1) + (n+1) + \dots + (n+1)}_{n/2 \text{ terms}} = \boxed{\frac{n}{2} \cdot (n+1)} \quad \checkmark$$

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of these, \boxed{n} are translational (in 2D: 2; in 3D: 3). \checkmark

The rotational ones are then $\frac{n}{2} (n+1) - n = n \left(\frac{n+1-1}{2} \right) = \boxed{\frac{n(n-1)}{2}} \quad \checkmark$

②

I will use Grubler's formula: $\text{DOF} = m(N-1) - \sum_{i=1}^J (m-f_i)$

The four-legged walking robot is assumed to move in a 3D environment, so $m=6$.

Each leg has 3 links and the body is another link. We always count the ground (I assume the robot is resting on its body if no legs are on the ground; it cannot be floating). So, $N=14$. Each U joint has 2 dof and each R has 1 dof. There are $J=12$ links. There are $2+1$ dof for each leg and there are four legs.

$$\text{So, } \text{DOF} = m(N-1-J) + \sum_{i=1}^J f_i = 6 \cdot (14-1-12) + 16 = \boxed{22 \text{ dof}} \quad \checkmark$$

One leg on GND One constraint is added to the system. The leg that rests on the ground - I assume - can still slip on the ground & use its freedom as a revolute joint. However, it must remain in contact with the ground and that is an added constraint. So, now the robot has $22-1 = \boxed{21 \text{ dof}}$ (The constraint equation has explicitly to do with the vertical (Z -axis) difference from the ground and that leg - and has to be zero)

Two legs

Similar argument; there is an added constraint, so $\boxed{20 \text{ dof}}$ \checkmark

Three/four legs

There are $\boxed{19 \text{ dof}}$ for 3 legs & $\boxed{18 \text{ dof}}$ for 4 legs on the ground

$$\text{ii) } \text{DOF} = m(N-1-J) + \sum f_i = 6 \cdot (14-1-12) + 20 = 26 \text{ dof}$$

Each leg on the ground results in one loss dof. (25 dof for 1 leg, 23 dof for 3 legs, 24 dof for 2, 22 dof for all 4)

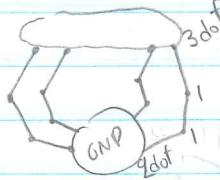
iii) I treat the palm as ground, as it has 0 degrees of freedom.

Each finger has 9+1+1 dof and the object the hand is grasping has 6dof. Each fingertip now becomes a joint between the object and the finger and has 3dof (starting from 6dof, the tips must touch the object (1st constraint - e.g. z-axis) and cannot slide (2 constraints, e.g. neither in the x nor y direction). Each finger has 3 links, so $N = 3 \cdot 4 + 1$ (the object) + 1 (the ground/base)

$$N=14. \text{ There are } J=16 \text{ joints so } \text{dof} = 6 \cdot (14-1-16) + \sum f_i = -18 + 4 \cdot 7 = 10 \text{ dof. Or}$$

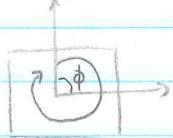
$$m(N-1) - \text{constraints} = 6 \cdot (14-1) - 4 \cdot (3+5+5+4) = 6 \cdot 13 - 4 \cdot 17 = 78 - 68 = 10 \text{ dof [a joint with 1dof has 5 constraints, 2dof - 4 constraints and so on]}$$

dof + constraints = 6 at all times



- i) If the fingertips can roll, they can effectively move "slip" by rotating around in a way such that the same point is touching the object but at a different location. Thus, two constraints are freed and only the contact (let's say constraint in the z-axis difference) remains. Now, the fingertips have 5dof. So, dof = $6 \cdot (14-1-16) + 4 \cdot 9 = -18 + 36 = 18 \text{ dof}$ ✓ 10/10

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- i) The chassis can move anywhere on the plane (x & y direction). The car can also turn (rotation around the axis \perp to the plane - z-axis). So, the C-space is $S^1 \times E^2$ or - using the representations to express it $SO^2 \times \mathbb{R}^2$.
- ii) The back wheels do not steer; they can only rotate and at different speeds (think of the car turning; the outer wheel moves faster). The front wheels are exactly as the back wheels, but they can also steer (by the same amount each!). So, they both have 2dof, but one constraint; given the steering angle of one, I know the steering angle of the other, too. I am unsure how to include that constraint. I will write it as if one of the front wheels has only one dof. So,

S^2 for one front wheel; the one that has a steering dof. too.

$$(S^1 \times E^2) \times (S^1 \times S^1 \times S^1 \times S^2) = (S^1 \times E^2) \times (T^3 \times S^2) \quad \checkmark$$

- iii) Imagine the infinite plane rolling up & becoming a sphere; all else is the same. The C-space of a sphere is S^2 (instead of \mathbb{R}^2 for a plane). So, the C-space is: $(S^2 \times S^1) \times (T^3 \times S^2)$ ✓

- iv) Same set-up as i) - including the C-space of the wheels. The C-space of the robot arm is $S^1 \times S^1 \times E^1[a,b] \times S^1$ where $[a,b]$ denotes the range over which the prismatic joint can move. So, all together the C-space is $(S^1 \times E^2) \times (T^3 \times S^2) \times (T^3 \times E^1[a,b])$, where $T^3 = S^1 \times S^1 \times S^1$

R^n is the representation of the E^n space.

- v) Same setup, but now all joints (I assume only joints of the arm) have limits.

$$\text{So, the C-space is: } [(S^1 \times E^2) \times (T^3 \times S^2) \times S^1[\phi_0, \phi_f] \times S^1[\theta_0, \theta_f] \times S^1[\omega_0, \omega_f] \times E^1[a,b]]^4$$

- vi) A free-flying spacecraft has the C-space of a rigid body: $\mathbb{R}^3 \times SO(3)$. The 6R arm has a C-space of

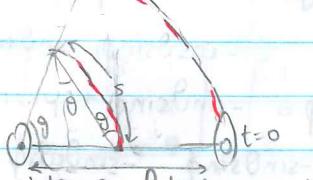
(continue 3)

$S \times S' \times S' \times S' \times S' \times S' = T^6$ where S' is the C-space of a revolute joint. In total, the C-space is: $[R^3 \times SO(3)] \times T^6$

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4)

$\dot{\theta} = \dot{\theta}_1 - \dot{\theta}_2$ --- movement

i)



$\dot{\phi}_1 = w_1, \dot{\phi}_2 = w_2$. If a wheel of radius R rotates by $\dot{\phi}$, it covers a distance $R\dot{\phi}$. In the drawing on the left, the left wheel does not move & the right wheel travels $|R\cdot\dot{\phi}_2| = |\theta \cdot D|$ (where D is the distance θ between the front wheel). Then, $\theta = -R\cdot\dot{\phi}_2$ (if only the right wheel moves). If only the left wheel moves, $\theta = +R\cdot\dot{\phi}_1$. So, if both wheels move $\theta = \frac{R}{D}(\dot{\phi}_1 - \dot{\phi}_2)$ and $\dot{\theta} = \frac{R}{D}(\dot{\phi}_1 - \dot{\phi}_2) = \frac{R}{D}(w_1 - w_2) = \dot{\theta}$. It makes sense; there is no $\dot{\theta}$ if both wheels turn at the same rate.

The point halfway between the wheels moves along s (see figure). Its x and y displacement is the average of the corresponding displacements of the front wheels. Same argument for the time derivative (just differentiate the relationship). So, $\dot{x} = \frac{x_{\text{left}} + x_{\text{right}}}{2}$ and $\dot{y} = \frac{y_{\text{left}} + y_{\text{right}}}{2}$. The velocity of each wheel is $R\cdot\dot{\phi}_i$. To find the x - and y -components, we multiply by $\sin\theta$ and $\cos\theta$ respectively.

If the car moves forward, $\theta = 0$, $\sin\theta = 0$ and $\dot{x} = 0$; it works.

$$\begin{aligned} \dot{s}, \dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}(R\dot{\phi}_1 + R\dot{\phi}_2)\sin\theta \\ \frac{1}{2}(R\dot{\phi}_1 + R\dot{\phi}_2)\cos\theta \\ \frac{R}{D}(\dot{\phi}_1 - \dot{\phi}_2) \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} \frac{R}{2}\sin\theta \\ \frac{R}{2}\cos\theta \\ \frac{R}{D} \\ 0 \\ 0 \end{bmatrix} w_1 + \begin{bmatrix} \frac{R}{2}\sin\theta \\ \frac{R}{2}\cos\theta \\ -\frac{R}{D} \\ 0 \\ 0 \end{bmatrix} w_2 \quad \checkmark \end{aligned}$$

ii) I move everything to the left:

$$\begin{bmatrix} \dot{x} - R_1 \sin\theta w_1 - R_2 \sin\theta w_2 \\ \dot{y} - R_1 \cos\theta w_1 - R_2 \cos\theta w_2 \\ \dot{\theta} - R_1 w_1 + R_2 w_2 \\ \dot{\phi}_1 - \dot{\phi}_1 \\ \dot{\phi}_2 - \dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which I can rewrite as:

by observation

$$\begin{bmatrix} 1 & 0 & 0 & -R_1 \sin\theta & -R_2 \sin\theta \\ 0 & 1 & 0 & -R_1 \cos\theta & -R_2 \cos\theta \\ 0 & 0 & 1 & -R_1/D & R_2/D \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = 0$$

iii) The constraints are nonholonomic; the car cannot instantly move to the right or left (velocity constraint) but can do so effectively through a series of movements; consider parallel parking $\dot{\theta} \uparrow = \dot{\theta} \rightarrow$

5)

My palm is connected to the rest of my body through my wrist. It can effectively move in all directions ($\hat{x}, \hat{y}, \hat{z}$ -translations & three rotational degrees). Thus, its task space is $SO(3) \times R^3$ and its dimensions are 6 (6 dof).

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6) From p.45 of the lecture notes (Def. 3.2) a matrix R in $SO(2)$ can be written as: $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

i) Let, $Q = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ another $SO(2)$ matrix - different angle of rotation $\phi \neq \theta$.

$$R \cdot Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos\theta \cos\phi - \sin\theta \sin\phi & -\cos\theta \sin\phi - \sin\theta \cos\phi \\ \sin\theta \cos\phi + \cos\theta \sin\phi & -\sin\theta \sin\phi + \cos\theta \cos\phi \end{bmatrix}$$

$$Q \cdot R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\phi \cos\theta - \sin\phi \sin\theta & -\sin\phi \cos\theta - \cos\phi \sin\theta \\ \sin\phi \cos\theta + \cos\phi \sin\theta & -\sin\phi \sin\theta + \cos\phi \cos\theta \end{bmatrix}$$

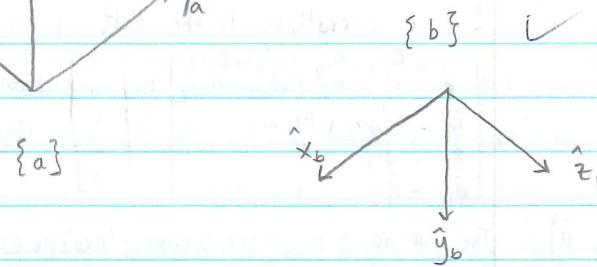
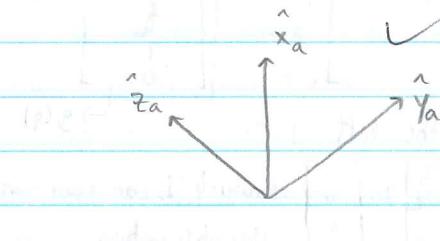
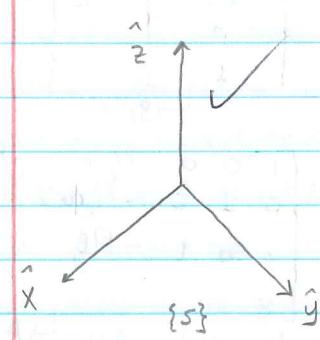
We see that two random $SO(2)$ matrices commute, since $R \cdot Q = Q \cdot R$, $R, Q \in SO(2)$

ii) Matrices generally do not commute and all it takes is a counterexample to show they do not commute. I use two $SO(3)$ matrices from class (it can be shown they satisfy $\det R=1$ & $R^T R = I$ but I will not show it here). Let $R_1 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ $R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (from Oct. 7th 2014)

$$R_1 \cdot R_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{Then, } R_2 \cdot R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(Clearly, $R_1 \cdot R_2 \neq R_2 \cdot R_1$, so $SO(3)$ matrices do not commute.) ✓

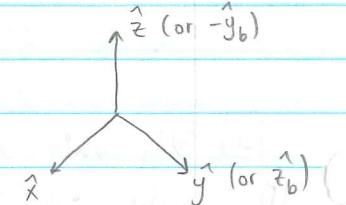
7)
i)



To find the z -direction, I use the right hand rule (as coordinate frames are right handed)

ii) $R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$ ✓ $R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ ✓

iii) $R_{sb}^{-1} = R_{sb}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ ✓
 ↴ y in terms of z_b



(continue 2)

$$iv) R_{ab} = R_{as} \cdot R_{sb} = R_{sa}^{-1} \cdot R_{sb} = R_{sa}^T \cdot R_{sb} \text{ so } R_{ab} = R_{sa}^T \cdot R_{sb}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$v) R = R_{sb}$$

$$R_1 = R_{sa} \cdot R = R_{sa} \cdot R_{sb} \quad \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$$

R_1 corresponds to rotating R_{sa} about the \hat{x}_a axis (multiplied on the right - body frame)

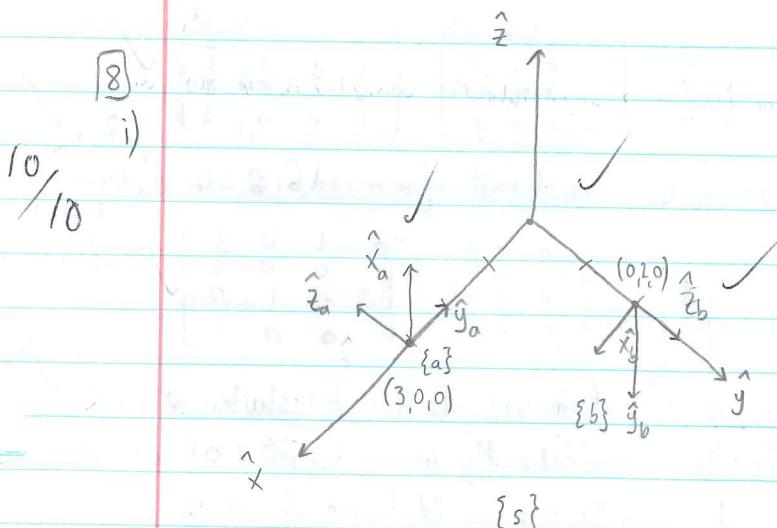
$$R_2 = R R_{sa} = R_{sb} \cdot R_{sa} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

R_2 corresponds to rotating R_{sa} about the world-fixed axis \hat{x} (multiplied on the left)

$$vi) R_{sb} \cdot p_b = p_s' \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \checkmark$$

$$vii) p' = R_{sb} \cdot p_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \checkmark \text{ Transform of the location of the point without changing the reference frame} \checkmark$$

$$viii) p'' = R_{sb}^T \cdot p_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \checkmark \text{: Change coordinates from } \{s\} \text{ to } \{b\} \text{ No transform of location} \checkmark$$



$$ii) R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \quad R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$iii) T_{sb} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{as } T_{sb} = \begin{bmatrix} R_{sb} & p \\ 0 & 1 \end{bmatrix}$$

$$\text{where } T_{sb}^{-1} = \begin{bmatrix} R_{sb}^T & -R_{sb}^T p \\ 0 & 1 \end{bmatrix}$$

$$-R_{sb}^T p = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\text{so } T_{sb}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$iv) T_{ab} = T_{as} \cdot T_{sb} = T_{sa}^{-1} \cdot T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{where } -R_{sa}^T \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\text{so } T_{ab} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$v) T = T_{sb}$$

$$T_1 = T_{sa} \cdot T = T_{sa} \cdot T_{sb} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_1 corresponds to a body-fixed $\{a\}$ -frame transformation [we moved by 2 in the \hat{y}_a direction to go to point \vec{r}_{s1}]

$$T_2 = T \cdot T_{sa} = T_{sb} \cdot T_{sa} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_2 corresponds to a world-fixed transformation of T_{sb} . We started at point $(3, 0, 0)$ & performed a T_{sb} transformation (translationally moving by $\vec{p}(0, 2, 0)$) and we ended at $(3, 2, 0)$. So, we moved by 2 in the world-frame \hat{y} -axis.

(continue 3)

$$vi) p_s = T_{sb} \cdot p_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{5}{2} \\ 1 \end{bmatrix} \text{ so } p_s = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \checkmark$$

$$vii) p' = T_{sb} \cdot p_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -2 \\ 1 \end{bmatrix} \text{ so } p' = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} \checkmark$$

$$p'' = T_{sb} \cdot p_s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix} \text{ so } p'' = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} \checkmark$$

$\left[\begin{smallmatrix} 0 \\ 2 \\ 3 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 1 \\ -3 \\ 0 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \right]$ I believe, for p' , p_s is actually p_b . Then, starting from $\{s\}$, we translate by $\left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \right]_{\{s\}}$ or $\left[\begin{smallmatrix} 1 \\ 5 \\ -2 \end{smallmatrix} \right]_{\{s\}}$ and we end at $\left[\begin{smallmatrix} 1 \\ 5 \\ -2 \end{smallmatrix} \right]_{\{s\}}$. So, we have performed a translation without reference frame change. \checkmark

For p'' , we changed the coordinates from $\{s\}$ to $\{b\}$ (if we are at the center of $\{b\}$ we are at $(0, 2, 0)$ (seen from the $\{s\}$ frame). Then, we need to move by 1 towards the \hat{x} -axis ($\{s\}$ frame) - also \hat{x}_b -axis in $\{b\}$ frame - and by 3 towards the \hat{z} -axis ($\{s\}$ frame), which is -3 in the \hat{y}_b -axis in $\{b\}$. So, now point $p_s = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \right]$, looked from the center of $\{b\}$ is $\left[\begin{smallmatrix} 2 \\ 3 \\ 3 \end{smallmatrix} \right] - \left[\begin{smallmatrix} 0 \\ 2 \\ 0 \end{smallmatrix} \right] = \left[\begin{smallmatrix} 1 \\ 1 \\ 3 \end{smallmatrix} \right]$ in $\{s\}$ coordinates.

Transformation mapping between $\{s\}$ & $\{b\}$:

9) I followed the book's example (3.70 eq)

$$\text{tr } R = r_{11} + r_{22} + r_{33} = 1 + 2 \cos 0$$

Cases

(i) $R \neq I$, go to next case

(ii) $\text{tr } R = 0 \neq -1$ go to next case

$$(iii) \theta = \cos^{-1} \left(\frac{\text{tr } R - 1}{2} \right) \in [0, \pi] \Rightarrow \theta = \cos^{-1} \left(-\frac{1}{2} \right) \Rightarrow \boxed{\theta = \frac{2\pi}{3}} \text{ and } [\omega] = \frac{1}{2 \sin \theta} (R - R^T)$$

$$\text{where } R - R^T = - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \text{ where } [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$$\frac{1}{2 \sin \theta} = \frac{1}{\sqrt{3}}$$

so I read the values off $[\omega]$: $\omega_1 = \frac{1}{\sqrt{3}}$, $\omega_2 = -\frac{1}{\sqrt{3}}$, $\omega_3 = \frac{1}{\sqrt{3}}$

$$\text{so } \omega = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \theta = \frac{2\pi}{3} \text{ so } \boxed{\omega \cdot \theta = \frac{2\pi}{3} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}} \checkmark$$

10/10

[10] To convert from angular velocity in the E_B frame to angular velocity in the E_S frame, we use the property of "subscript cancellation": $\omega_i = R_{ij} \omega_j$
 So, to get ω_S I need to multiply ω_B with R_{SB} (on the left)

Then:

$$\omega_S = R_{SB} \cdot \omega_B = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}} = \omega_S \quad \checkmark \quad 10/10$$