Where we are:

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Important concepts, symbols, and equations

• $SO(3)$ is a curved 3-dimensional space, but the feasible velocities at any point of $SO(3)$ form a flat 3-dimensional **vector space** (the “tangent space”).

Another example: the tangent space at a point of $S^2$.

• Any rotational velocity can be expressed as an **angular velocity** $\omega \in \mathbb{R}^3$, which can be considered the product of a unit axis (in $S^2$) and a speed (a scalar).
Important concepts, symbols, and equations (cont.)

• Given $p \in \mathbb{R}^3$ and $\omega$ defined in the same reference frame, $\dot{p} = \omega \times p$.

• Linear algebra notation: $\dot{p} = \omega \times p = [\omega] p$, where

$$
[x] = \begin{bmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0 \\
\end{bmatrix} \in so(3), \text{ the } 3 \times 3 \text{ real skew-symmetric matrices (satisfying } [x] = -[x]^T).$$

$so(3)$ describes the possible $\dot{R}$ when $R = I$, and it is called the Lie algebra of the Lie group $SO(3)$.
Important concepts, symbols, and equations (cont.)

- If $R_{sb} = [p_1 \ p_2 \ p_3]$, then $\dot{R}_{sb} = \begin{bmatrix} [\omega_s] p_1 & [\omega_s] p_2 & [\omega_s] p_3 \end{bmatrix} = [\omega] R_{sb}$.

- Expressing the angular velocity in a different frame:

$$\omega_b = R_{bs} \omega_s = R^{-1}_{sb} \omega_s = R^T_{sb} \omega_s \quad \omega_s = R_{sb} \omega_b$$

- The $so(3)$ representations:

$$[\omega_b] = R^{-1}_{sb} \dot{R} = R^T_{sb} \dot{R} \quad [\omega_s] = \dot{R} R^{-1}_{sb} = \dot{R} R^T_{sb}$$

- Exponential coordinate (axis-angle) representation of orientation: $\hat{\omega} \theta$
Important concepts, symbols, and equations (cont.)

• Scalar first-order linear diffeq:
  \[ \frac{dx(t)}{dt} = ax(t) \implies x(t) = e^{at}x_0 \]
  
  \[ e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots \]

• Vector first-order linear diffeq:
  \[ \dot{x}(t) = Ax(t) \implies x(t) = e^{At}x_0 \]
  
  \[ e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \]

  matrix exponential
Important concepts, symbols, and equations (cont.)

• Integrating an angular velocity

\[ \dot{p} = \hat{\omega} \times p = [\hat{\omega}]p \quad \Rightarrow \quad p(t) = e^{[\hat{\omega}]} p(0) \]

\[ p(\theta) = e^{[\hat{\omega}]\theta} p(0) \]

\[ \text{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta} = I + \sin \theta \, [\hat{\omega}] + (1 - \cos \theta)[\hat{\omega}]^2 \in SO(3) \]

Rodrigues’ formula

• Matrix exponential and matrix log:

\[ \exp : [\hat{\omega}]\theta \in so(3) \rightarrow R \in SO(3) \]

\[ \log : R \in SO(3) \rightarrow [\hat{\omega}]\theta \in so(3) \]
1) What rotation axis \( \mathbf{\hat{\omega}} \) and angle \( \Theta \) take \( \mathcal{E}_{13} \) to \( \mathcal{E}_{23} \) expressed in \( \mathcal{E}_{13} \) frame

\[ \mathbf{w} \in \mathbb{R}^3 \]

2) Write this as an angular velocity \( \mathbf{\omega} \), which, if executed for \( t = 1 \), achieves the rotation. (the exponential coordinates)

3) Write this as \( [\mathbf{w}] \in \mathfrak{so}(3) \), the skew-symmetric matrix form.
1) \( R_{sb} = \)

2) \( R_{1} = e^{i\frac{\omega}{2} T_{2}} = \)

3) \( R_{sb}' = R_{sb} R_{1} = \)

4) \( R_{sb}'' = R_{1} R_{sb} = \)