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Two-Dimensional Rigid-Body Collisions With Friction

This paper presents an analysis of a two-dimensional rigid-body collision with dry friction. We use Routh's graphical method to describe an impact process and to determine the frictional impulse. We classify the possible modes of impact, and derive analytical expressions for impulse, using both Poisson's and Newton's models of restitution. We also address a new class of impacts, tangential impact, *with zero initial approach velocity. Some methods for rigid-body impact violate energy conservation principles, yielding solutions that increase system energy during an impact. To avoid such anomalies, we show that Poisson's hypothesis should be used, rather than Newton's law of restitution. In addition, correct identification of the contact mode of impact is essential.*

1 Introduction

Although planar rigid-body impact has been studied for centuries, and is discussed in almost all dynamics texts, there are still unresolved difficulties:

- There are two competing laws governing the coefficients of restitution: *Newton's law* and *Poisson's hypothesis*. When do they give the same system behavior? Is one law preferable to the other?
- Some methods, for instance in Whittaker (1944), can result in an increase in total energy, violating basic energy conservation principles (Keller 1986; Brach 1984). How can we avoid such anomalies?

This paper resolves these difficulties by adopting Poisson's hypothesis of restitution and by using Routh's method (Routh 1860) to determine the resultant impulsive forces. The Routh-Poisson analysis gives an impulse consistent with Coulomb's law, without an increase in total energy. An interesting dividend is that the Routh-Poisson analysis admits a new class of impact, called *tangential impact*, defined as an impact with zero initial approach velocity.

Routh's method is a simple graphical technique for analyzing frictional impact in the plane. Using Coulomb's law of dry friction, and either Newton or Poisson restitution, Routh's method readily predicts the total impulse. We can also use Routh's method to distinguish several different types of contact, to identify cases where relative sliding either ceases or reverses, and to identify the cases where Newton and Poisson restitution differ in their predictions. Although this geometrical approach is restricted to planar problems, Keller (1986) develops an analytical method that extends the fundamental concepts to three-dimensional problems. Han and Gilmore (1989) apply the same approach to multiple-contact impact.

The choice between Newton's law of restitution, and Poisson's hypothesis, is particularly important. Newton prescribes the final normal velocity, while Poisson prescribes the normal forces applied during restitution, a difference which leads Kilmister and Reeve (1966) to argue that Poisson's hypothesis is philosophically superior to Newton's law. In the simplest cases, the two methods give identical results, but generally they do not. Although Newton's law of restitution is the more commonly applied method, we show that the violations of energy principles can be attributed to Newton's law of restitution.

Section 2 reviews the classical impact model of collision and the definitions of restitution and friction. Section 3 describes the equations of motion and introduces Routh's graphical technique. Sections 3.4 and 3.5 identify the different classes of impact and derive solutions for each class. Sections 4 and 5 derive expressions for system energy change and compare Newton's law of restitution and Poisson's hypothesis. Section 6 presents some examples using both Routh-Poisson and Routh-Newton. Finally, Section 7 contains a few concluding remarks. Some of results were previously presented in (Wang, 1986; Wang and Mason, 1987; Mason and Wang, 1988).

2 Rigid-Body Model of Collision

The sudden, short-term encounter between two colliding bodies is a very complicated event. The major characteristics are the very brief duration and the large magnitudes of the forces generated. Other phenomena include vibration waves propagating through the bodies, local deformations produced in the vicinity of the contact area, and frictional and plastic dissipation of mechanical energy. The complexity of the process leads to serious difficulties in the mathematical analysis of the problem. By introducing the rigid-body assumption and Coulomb's law, we simplify the analysis while retaining a fair approximation of a significant class of real systems.

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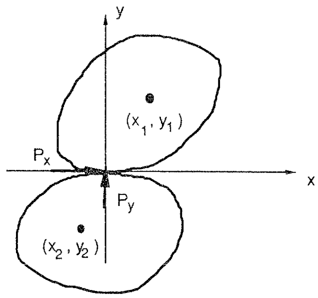

Fig. 1 Two colliding rigid bodies in a plane. The normal impulse and tangential impulse acting on the body 1 are shown as P_y and P_x.

For the collision of two rigid bodies (Fig. 1), the primary simplifying assumption is a postulated deformation history. This deformation history is assumed to consist of two periods: the period of compression and the period of restitution. The compression period extends from the instant of contact to the point of maximum compression, when the approach velocity becomes zero. The period of restitution then begins, lasting to the instant of separation. The time interval of the contact is assumed to be very small and the interaction forces are high. These postulates permit some further assumptions. (1) The collision process is instantaneous, and linear and angular velocities of the bodies have discontinuous changes. (2) Interactive forces are impulsive, and all other finite forces are negligible. (3) No displacements occur during the collision.

2.1 Coefficient of Restitution. During the brief period of contact, a normal force F acts along the common normal between the two bodies.[1] Since the contact duration is sufficiently small, the contact force may be represented by Dirac's delta function.

$$P = \lim_{\Delta t \to 0} \int F(t)dt. \tag{1}$$

It is called *impulse* and is defined to be finite.

The magnitude of the normal impulse consists of two parts, P_c and P_r, corresponding to the periods of compression and restitution, respectively. The total impulse is the sum of the two parts

$$P_y = P_c + P_r. \tag{2}$$

If we adopt *Poisson's hypothesis* (Beer and Johnston, 1984), it is further postulated that the ratio of P_r to P_c is determined,

$$e = \frac{P_r}{P_c}. \tag{3}$$

This constant e is called the *coefficient of restitution*, and is assumed to depend solely on the materials of the bodies (Goldsmith, 1960). The coefficient describes the degree of plasticity of the collision, and its value is always between zero and one. When $e = 0$, the impact is said to be perfectly plastic; when $e = 1$, it is said to be perfectly elastic.

Poisson's hypothesis immediately suggests a model based on a hysteretic spring or other passive elements. The coefficient of restitution may, however, be put in another form, known as *Newton's law of restitution*, which cannot be modeled in this way. Newton's law of restitution states

$$e = -\frac{C^+}{C^-} \tag{4}$$

where C^- and C^+ are the normal components of relative velocity at the contact point before and after the collision, respectively.

Both Poisson's hypothesis and Newton's law have been adopted by the scientific community to describe the energy dissipation. However, they do not in general produce consistent solutions. In this paper, we discuss both definitions and their solutions of impact. Section 5 shows that Newton's law can lead to violation of energy conservation.

3 The Two-Dimensional Collision Problem

This section analyzes the process of two planar rigid bodies with friction. First, we present Routh's method. Then, we use Routh's method to classify the different kinds of impact and derive solutions for each class.

3.1 Equations of Motion. When two bodies collide, impulses in the normal direction P_y and in the tangential direction P_x at the contact point are produced. These impulses will change the object's motions. In the coordinate system shown in Fig. 1, the initial translational and rotational velocity components of the first object are $\dot{x}_{1o}$, $\dot{y}_{1o}$, and $\dot{\theta}_{1o}$. The origin of the coordinates is chosen at the point of contact. The coordinate axes x and y are in the directions tangential and normal to the contact surfaces. At any instant during the impact, the motion of the object is governed by the linear and angular impulse-momentum laws, which provide the following relations:

$$m_1(\dot{x}_1 - \dot{x}_{1o}) = P_x \tag{5}$$

$$m_1(\dot{y}_1 - \dot{y}_{1o}) = P_y \tag{6}$$

$$m_1\rho_1^2(\dot{\theta}_1 - \dot{\theta}_{1o}) = P_x y_1 - P_y x_1 \tag{7}$$

where m_1 is the mass, ρ_1 is the radius of gyration of inertia, and x_1 and y_1 are the coordinates of the center of mass for the first object.

The velocity of the point of contact on the first object consists of two components, $\dot{x}_{1c}$ and $\dot{y}_{1c}$. These two components are given by

$$\dot{x}_{1c} = \dot{x}_1 + \dot{\theta}_1 y_1 \tag{8}$$

$$\dot{y}_{1c} = \dot{y}_1 - \dot{\theta}_1 x_1. \tag{9}$$

Similarly, we obtain the dynamic equations for the second object

$$m_2(\dot{x}_2 - \dot{x}_{2o}) = -P_x \tag{10}$$

$$m_2(\dot{y}_2 - \dot{y}_{2o}) = -P_y \tag{11}$$

$$m_2\rho_2^2(\dot{\theta}_2 - \dot{\theta}_{2o}) = -P_x y_2 + P_y x_2 \tag{12}$$

$$\dot{x}_{2c} = \dot{x}_2 + \dot{\theta}_2 y_2 \tag{13}$$

$$\dot{y}_{2c} = \dot{y}_2 - \dot{\theta}_2 x_2 \tag{14}$$

where m_2 is the mass, ρ_2 is the radius of gyration of inertia, and x_2 and y_2 are the coordinates of the center of mass for the second object.

From Eqs. (8), (9), (13), and (14), the tangential component of relative velocity of the points in contact is called *sliding velocity* and is given as

$$S = \dot{x}_{1c} - \dot{x}_{2c}$$
$$= (\dot{x}_1 + \dot{\theta}_1 y_1) - (\dot{x}_2 + \dot{\theta}_2 y_2) \tag{15}$$

and the normal component of relative velocity is called *compression velocity* and is given as

$$C = \dot{y}_{1c} - \dot{y}_{2c}$$
$$= (\dot{y}_1 - \dot{\theta}_1 x_1) - (\dot{y}_2 - \dot{\theta}_2 x_2). \tag{16}$$

Substituting the dynamic Eqs. (5)-(7) and (10)-(12) into these kinematic equations, we find that

$$S = S_o + B_1 P_x - B_3 P_y \tag{17}$$

$$C = C_o - B_3 P_x + B_2 P_y \tag{18}$$

[1]If one of the contact points is a vertex, the common normal is defined as the normal of the other body's surface. We do not consider the case of two vertices in contact.

where

$$B_1=\frac{1}{m_1}+\frac{1}{m_2}+\frac{y_1^2}{m_1\rho_1^2}+\frac{y_2^2}{m_2\rho_2^2} \tag{19}$$

$$B_2=\frac{1}{m_1}+\frac{1}{m_2}+\frac{x_1^2}{m_1\rho_1^2}+\frac{x_2^2}{m_2\rho_2^2} \tag{20}$$

$$B_3=\frac{x_1y_1}{m_1\rho_1^2}-\frac{x_2y_2}{m_2\rho_2^2} \tag{21}$$

and

$$\begin{aligned} S_o &= \dot{x}_{1co}-\dot{x}_{2co} \\ &= (\dot{x}_{1o}+\dot{\theta}_{1o}y_1)-(\dot{x}_{2o}+\dot{\theta}_{2o}y_2) \end{aligned} \tag{22}$$

$$\begin{aligned} C_o &= \dot{y}_{1co}-\dot{y}_{2co} \\ &= (\dot{y}_{1o}-\dot{\theta}_{1o}x_1)-(\dot{y}_{2o}-\dot{\theta}_{2o}x_2). \end{aligned} \tag{23}$$

Note that S_o and C_o are the initial values of sliding and compression velocities. B_1, B_2, and B_3 are constants, dependent on the geometry and mass properties of the system, with B_1 and B_2 always positive. In all cases, $B_1B_2 > B_3^2$, which will be useful in later sections.

3.2 Restitution and Friction. The algebraic Eqs. (17) and (18) give the relative velocity (S, C) as a function of total accumulated impulse (P_x, P_y). Routh's method also requires that we express the laws governing restitution and friction in terms of the total accumulated impulse (P_x, P_y).

3.2.1 Coefficient of Restitution. By assumption, object deformation consists of two phases: compression and restitution. At the end of the compression phase, the normal component of the relative velocity of the points in contact is zero ($C = 0$). Substituting Eq. (18), we obtain a linear relationship between the impulse components at maximum compression:

$$C_o-B_3P_x+B_2P_y=0. \tag{24}$$

In the (P_x, P_y) space, this equation defines a straight line called the *line of maximum compression*.

After the point of maximum compression, the restitution phase begins, lasting to the end of the collision. Under Newton's law, the collision ends when the normal velocity C is $-e$ times the initial normal velocity C_0. That is

$$e=-\frac{C(t_f)}{C(t_o)} \tag{25}$$

where t_o is the initial collision time and t_f is the termination time. Substituting this into Eq. (24), we obtain

$$(1+e)C_o-B_3P_x+B_2P_y=0 \quad (t=t_f). \tag{26}$$

Again, we obtain a line in the (P_x, P_y) space, called the *line of termination*. When using Newton's law of restitution, the total impulse will always fall on the line of termination.

But under Poisson's hypothesis, the collision ends when the total normal impulse P_y is $(1+e)$ times that value of P_y at maximum compression. That is

$$\frac{P_y(t_f)}{P_y(t_c)}=1+e \tag{27}$$

where t_c denotes the instant of maximum compression. Unlike Newtonian restitution, Poisson's hypothesis does not yield a line of termination.

3.2.2 Coefficient of Friction. Friction causes an impulsive force in the tangential direction at the contact point. We adopt Coulomb's law to determine the force of dry friction. The law states that the magnitude of the (tangential) frictional force F_x depends only on the magnitude of the normal force F_y and the materials in contact, and its direction is always opposite that of relative tangential motion. This law is commonly expressed as

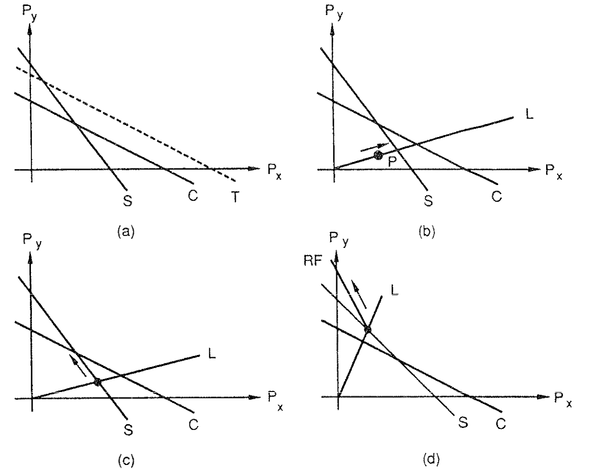

Fig. 2 Impact process diagram. The lines of sticking, lines of maximum compression, and lines of termination are labeled, respectively, with S, C, and T. The lines of limiting friction and the line of reversed limiting friction are labeled with L and RF, respectively. The point P is the representative point.

$$|F_x| \le \mu F_y \tag{28}$$

where μ is the *coefficient of friction* and is an empirical constant. In this paper, we do not distinguish between static and dynamic friction and we take the values for corresponding noncollision processes.

Coulomb's law includes two different cases: sticking and sliding. Since differential impulse is force, these cases are expressed as

$$|dP_x| < \mu dP_y \text{ for sticking}$$

$$|dP_x| = \mu dP_y \text{ for sliding.}$$

In the sticking case, the tangential component of relative velocity vanishes ($S = 0$). Again we can substitute Eq. (17), obtaining a linear relation between the components of impulse (P_x, P_y),

$$S_o+B_1P_x-B_3P_y=0. \tag{29}$$

This gives a straight line in the (P_x, P_y) space, called the *line of sticking*.

3.3 Impact Process Diagram. To solve an impact problem, we employ Routh's graphical technique to determine the total impulse. Figure 2 shows an example. We construct *impulse space* with coordinate axes P_x and P_y, and plot the accumulating impulse P. When the impact begins, P is at the origin. During the impact, the normal impulse P_y increases monotonically until the restitution law, which could be either Poisson or Newton, says that the impact is finished.

P_x also accumulates, in accordance with Coulomb's law. Assuming initial sliding, the impulse increases along a *line of limiting friction* (Fig. 2(b)) satisfying Coulomb's law:

$$P_x=-\mu s P_y \tag{30}$$

where s is the sign of the initial sliding velocity S_o,

$$s=\frac{S_o}{|S_o|} \text{ if } S_o \neq 0.$$

If the point reaches the line of sticking, the sliding will end and the frictional impulse will exhibit a change. There are two possibilities:

1 If the friction necessary to prevent sliding is less than the limiting friction, the point P will follow the line of sticking until the process terminates (Fig. 2(c)).

2 If the limiting friction is too small to prevent sliding, then P will cross the line of sticking, and the tangential force will

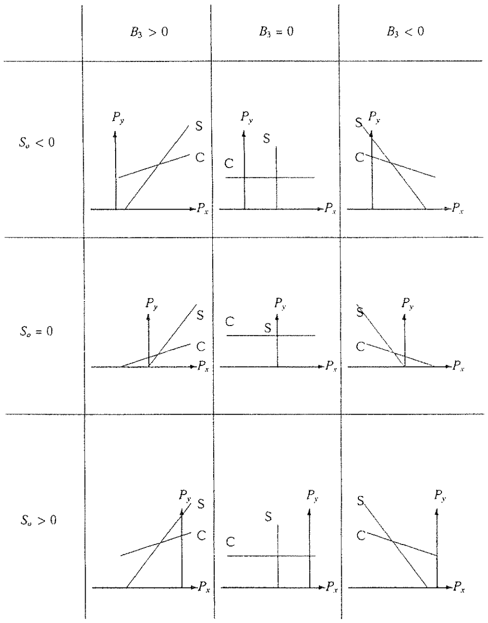

Fig. 3 All possible cases of meeting impact in the (P_x, P_y) space. The lines of sticking and maximum compression are labeled S and C.

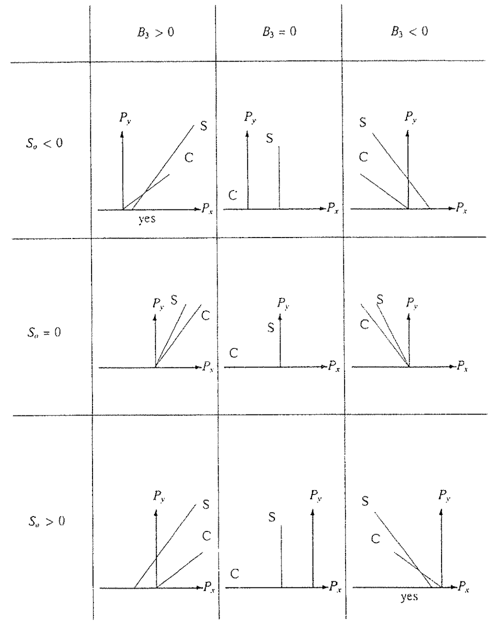

Fig. 4 There are two possible cases of tangential impact in the (P_x, P_y) space, which are labeled "yes." The lines of sticking and maximum compression are labeled S and C.

change sign, so that P now travels along the *line of reversed limiting friction* given by (Fig. 2 (*d*))

$$dP_x = \mu s dP_y.$$

Eventually, point P will cross the line of maximum compression. A perfectly plastic collision terminates at that point. By Poisson's hypothesis, a perfectly elastic collision continues until the normal impulse P_y is doubled. Intermediate cases, with coefficients of restitution between 0 and 1, terminate when the normal impulse is $(1 + e)$ times the value of P_y obtained at maximum compression. By Newton's law of restitution, the collision terminates when the point P reaches the line of termination.

The entire process can be summarized in a few lines. To recapitulate:

1 P moves initially along the line of limiting friction.

2 If P reaches the line of sticking, $S = 0$, then P switches to either the line of sticking, or the line of reversed limiting friction, whichever is steeper.

3 Termination occurs when:

(*a*) Newton: P reaches the line of termination.

(*b*) Poisson: P_y reaches a value $(1 + e)$ times its value at the line of maximum compression.

This procedure solves the impact problem by constructing the total impulse, from which we can immediately determine the resulting body motions.

3.4 Classes of Impact and Contact Modes. Routh's procedure, described above, is a graphical solution of impact problems. It can also be used to derive an analytic solution of impact problems. In this section we identify the different cases that must be considered. In the following section we derive analytic solutions of impact for each case using both Poisson's and Newton's methods.

3.4.1 Direct and Oblique, Central and Eccentric, Tangential and Meeting Impacts. Collision problems are classified first by the locations of the line of sticking and the line of maximum compression, which depend primarily on the signs of B_3, S_0, and C_0. Figures 3 and 4 show all possible combinations of the linear relationships, and thereby classifies all possible impacts. Figure 3 takes the case $C_0 < 0$, and plots all nine combinations for the signs of B_2 and S_0. The rows indicate the direction of the sliding velocity while the columns indicate the impact configuration. There are two special classes. If the initial sliding velocity is zero ($S_0 = 0$), the impact is called a *direct impact*, represented by the middle row in the figure. If $B_3 = 0$, the impact is called a *generalized central impact*, represented by the middle column. (Central impact, where the body centers of mass lie on the contact normal, is subsumed by generalized central impact.) Impacts which are neither direct, nor generalized central impacts, are called *eccentric oblique impacts*.

Figure 4 shows a new class of impacts, which we will call *tangential impacts*, which can occur when $C_0 = 0$. Previous work has only considered collisions with finite approach velocities, $C_0 < 0$, which we might term *meeting impacts*. Perhaps this reflects a bias towards finite force solutions. Indeed, Kilmister and Reeve (1966) even adopt a *principle of constraints* stating:

> constraints shall be maintained by forces, so long as this is possible; otherwise, and only otherwise, by impulses.

However, there are problems with zero initial compression velocity for which only impulsive forces will maintain the kinematic constraints. An example is presented in Section 6. So, even if we adopt the principle of constraints, tangential collisions are sometimes the only solution available.

Only two of the cases shown in Fig. 4 yield feasible tangential impacts: $S_0 < 0$ and $B_3 > 0$; or $S_0 > 0$ and $B_3 < 0$. In these two cases, and with a large enough coefficient of friction, the line of limiting friction passes immediately below the line of maximum compression, yielding a compressive phase followed by a restitution phase as in ordinary meeting impacts.

It is also natural to consider admitting collisions with positive compression velocities $C_0 > 0$, which we might term *parting impacts*. It is possible to view the restitution phase of a meeting impact as a parting impact, but otherwise we see no necessity for admitting parting impacts.

3.4.2 Contact Modes. The contact mode can be determined by a few simple comparisons. We consider the case with $S_o < 0$ and $B_3 < 0$ (Fig. 5). Other cases are similar. In Fig. 5, the line of sticking intersects the line of maximum compression at point Q and the line of termination at point D. The

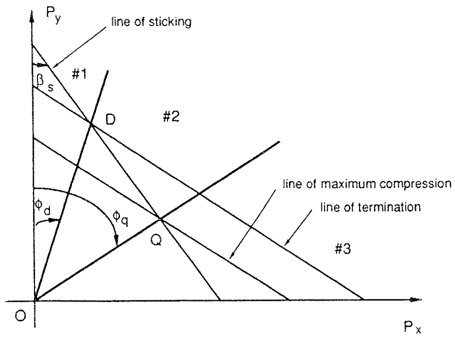

Fig. 5 Three regions in impulse space. In region 1, sticking never occurs; in region 2 and region 3, either a sticking or a reversed sliding contact occurs.

(P_x, P_y) space is divided into three regions by the lines OD and OQ. If the limiting friction line lies in region 1, the impact will be terminated before the representative point P reaches the line of sticking. The friction continues the limiting value throughout the process, so that the objects slide continuously. In this case Poisson and Newton give identical results, so the impact terminates at the line of termination.

If the limiting friction line lies in region 2, it reaches the line of maximum compression first, then reaches the line of sticking. If the limiting friction line lies in region 3, it reaches the line of sticking first. In either of regions 2 or 3, after intersecting with the line of sticking, it either continues sticking until termination or changes to reversal sliding.

These regions can be used to classify contact modes of impact. For an oblique impact, there are five contact modes: (1) sliding, (2) sticking in compression phase (C-sticking), (3) sticking in restitution phase (R-sticking), (4) reversed sliding in compression phase (C-reversed sliding), and (5) reversed sliding in restitution phase (R-reversed sliding). The classification depends on the values of μ, μ_d, μ_q, μ_s, P_d, and P_q given by

$$\mu = \tan\alpha \tag{31}$$

$$\mu_d = \tan\,\phi_d = \frac{(1+e)B_3C_o + B_2S_o}{(1+e)B_1C_o + B_3S_o} \tag{32}$$

$$\mu_q = \tan\,\phi_q = \frac{B_3C_o + B_2S_o}{B_1C_o + B_3S_o}. \tag{33}$$

$$\mu_s = \tan\,\beta_s = -\frac{B_3}{B_1} \tag{34}$$

$$P_d = (B_2 + s\mu B_3)sS_o \tag{35}$$

$$P_q = (\mu B_1 + sB_3)(-C_o). \tag{36}$$

The contact modes are summarized in Table 1. Note that the reversed sliding contact modes require that $S_oB_3 > 0$.

For a direct impact ($S_o = 0$), the line of sticking passes through the origin (Fig. 3). Region 2 vanishes and μ_d and μ_q have the same value as μ_s. Only two contact modes are possible, sticking (in compression phase) or sliding. Reversed sliding will never occur. As discussed by Routh, the representative point will follow either the line of limiting friction or the line of sticking throughout the entire process, depending on the following conditions:

$$\mu < |\mu_s| \text{ for sliding} \tag{37}$$

$$\mu > |\mu_s| \text{ for sticking.} \tag{38}$$

Wang and Mason (1987) and Han and Gilmore (1989) present similar results.

3.5 Analytical Solutions of Impulse. Once the contact mode is determined, we can solve for the impulses and object motions. If we define the sign function of initial sliding velocity S_o to be of value one when $S_o = 0$, the resulting impulses for both direct impact and oblique impact can be expressed in a unified form.

Table 1 Contact modes of impact, where μ is the friction coefficent and s is the sign function of S_o if $S_o \neq 0$

	$\mu > \lvert\mu_s\rvert$	$\mu < \lvert\mu_s\rvert$
$P_d > (1+e)P_q$	Sliding	
$P_q < P_d < (1+e)P_q$	R-Sticking	R-Reversed Sliding
$P_d < P_q$	C-Sticking	C-Reversed Sliding

For Poisson's method, the impulses are given by contact mode:

- sliding:

$$P_x = -s\mu P_y \tag{39}$$

$$P_y = -(1+e)\frac{C_o}{B_2 + s\mu B_3} \tag{40}$$

- C-sticking:

$$P_x = \frac{B_3P_y - S_o}{B_1} \tag{41}$$

$$P_y = -(1+e)\frac{B_1C_o + B_3S_o}{B_1B_2 - B_3^2} \tag{42}$$

- R-sticking:

$$P_x = \frac{B_3P_y - S_o}{B_1} \tag{43}$$

$$P_y = -(1+e)\frac{C_o}{B_2 + s\mu B_3} \tag{44}$$

- C-reversed sliding:

$$P_x = s\mu\left[P_y - \frac{2S_o}{B_3 + s\mu B_1}\right] \tag{45}$$

$$P_y = -\frac{1+e}{B_2 - s\mu B_3}\left[C_o + \frac{2s\mu B_3S_o}{B_3 + s\mu B_1}\right] \tag{46}$$

- R-reversed-sliding:

$$P_x = s\mu\left[P_y - \frac{2S_o}{B_3 + s\mu B_1}\right] \tag{47}$$

$$P_y = -(1+e)\frac{C_o}{B_2 + s\mu B_3} \tag{48}$$

where

$$s = \begin{cases} \dfrac{S_o}{|S_o|} & \text{if } S_o \neq 0 \\ 1 & \text{if } S_o = 0. \end{cases} \tag{49}$$

If we use Newton's law of restitution, the conditions for contact modes remain the same. However, whether sticking occurs in the restitution phase or in the compression phase does not affect the resulting impulses. The impulses are:

- sliding:

$$P_x = -s\mu P_y \tag{50}$$

$$P_y = -(1+e)\frac{C_o}{B_2 + s\mu B_3} \tag{51}$$

- sticking (C-sticking or R-sticking):

$$P_x = -\frac{B_2S_o + (1+e)C_oB_3}{B_1B_2 - B_3^2} \tag{52}$$

$$P_y = -\frac{B_3 S_o + (1+e) C_o B_1}{B_1 B_2 - B_3^2} \quad (53)$$

- reversed sliding (C-reversed sliding or R-reversed sliding):

$$P_x = s\mu \left[P_y - \frac{2S_o}{B_3 + s\mu B_1} \right] \quad (54)$$

$$P_y = -\frac{1}{B_2 - s\mu B_3} \left[(1+e) C_o + \frac{2s\mu B_3 S_o}{B_3 + s\mu B_1} \right] \quad (55)$$

where s is defined by Eq. (49).

Note that for sticking (in compression or in restitution) contact, the solutions are independent of the value of the coefficient of friction. As long as the friction is sufficient to prevent sliding, further increases do not matter. These expressions also appear in Wang (1989). Han and Gilmore (1989) present a similar analysis.

4 Energy Loss

Since some methods for rigid-body impact violate energy conservation principles, we develop expressions for the total energy loss during the impact. Due to the existence of friction and inelasticity, the system must lose some mechanical energy during the collision. The change in kinetic energy equals work done by the impulse. If T_1 and T_2 are the kinetic energies of body 1 and body 2, respectively, then the system energy change is (Routh 1860),

$$\Delta T = (T_1(t_f) + T_2(T_f)) - (T_1(t_o) + T_2(t_o))$$

$$= \frac{1}{2} [\mathbf{P}^T(\mathbf{V}_c + \mathbf{V}_{co})] \quad (56)$$

where, $\mathbf{P} = [P_x, P_y]^T$, $\mathbf{V}_c = [S\ C]^T$, and $\mathbf{V}_{co} = [S_o\ C_o]^T$. Substituting Eqs. (17) and (18) the energy change is

$$\Delta T = \frac{1}{2} (\mathbf{P}^T \mathbf{B} \mathbf{P} + 2\mathbf{V}_{co}^T \mathbf{P}) \quad (57)$$

where

$$\mathbf{B} = \begin{bmatrix} B_1 & -B_3 \\ -B_3 & B_2 \end{bmatrix}.$$

Conservation of energy requires that

$$\Delta T \le 0. \quad (58)$$

This gives a geometrical constraint in the impulse space: The total impulse must remain within an ellipse.

5 Poisson's Hypothesis Versus Newton's Law

By comparing the solutions of impulse presented in Section 3.5, we can identify the conditions under which Poisson's hypothesis and Newton's law give the same solution:

1 The collision is a direct impact, where the initial velocities of the contact points are directly along the common normal ($S_o = 0$) (Kilmister and Reeve (1966).
2 The collision is a generalized central impact ($B_3 = 0$).
3 The surfaces of the bodies are perfectly smooth and frictionless (Beer and Johnston, 1984).
4 The surfaces of the bodies are perfectly plastic ($e = 0$) (Wang, 1989).
5 The impact is of sliding contact, if friction between the bodies exists (Keller, 1986).

Now let us check energy conservation for the two models. We need examine only the perfect elastic case ($e = 1$), since any degree of plasticity will result in more energy loss. Under Poisson's hypothesis, an energy gain is impossible, which is verified by substituting the solutions of Section 3.5 into Eq. (56) (see Appendix). However, Newton's law of restitution sometimes produces energy gains. An example is given in the next section.

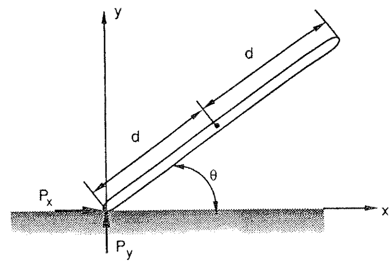

Fig. 6 A rigid rod colliding a frictional surface. In all cases, $m = 1$, length $d = 1/2$, $m\rho^2 = 1/12$, $\theta = 45$ deg, and initial angular velocity $\omega = 0$.

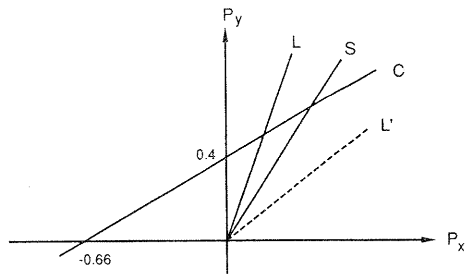

Fig. 7 Impact process diagram of the falling rod for case 1. Initial compression velocity $C_o = -1$ and initial sliding velocity $S_o = 0$. L and L' denote the lines of limiting friction for $\mu < 0.6$ and $\mu > 0.6$, respectively.

From both philosophical and practical points of view, Poisson's hypothesis is preferable to Newton's law of restitution. The philosophical reason, as argued by Kilmister and Reeve (1966), is that Poisson's hypothesis is expressed as a dynamic law, rather than as a kinematic constraint. The practical reason is that Poisson's hypothesis is consistent with energy conservation. This seems also to be consistent with Routh's original work where only Poisson's method is used.

6 Examples

This section illustrates our results with the example of a rod colliding with an immobile object (Fig. 6). This example has been used on many occasions to illustrate paradoxes in the mechanics of friction and impact (Goldsmith, 1960; Lotstedt, 1981; Brach, 1989; Erdmann, 1984). We assume point contact with Coulomb friction. The rod's initial orientation is $\theta = 45$ deg, and the initial angular velocity is zero $\omega(t_o) = 0$. The rod has unit mass and unit length ($m_1 = 1$, $\rho_1^2 = 1/12$). Note that $m_2 \to \infty$ and $m_2\rho_2^2 \to \infty$ and $B_1 = 2.5$, $B_2 = 2.5$, $B_3 = 1.5$, and $\mu_s = -0.6$.

By varying the initial velocity, we obtain four cases that illustrate the results of the paper.

Case 1: Direct Impact ($C_o = -1$ and $S_o = 0$). Since $S_o = 0$, this is a direct impact. The impact process diagram is shown in Fig. 7. From Eqs. (37) and (38), we find that if $\mu < 0.6$, sliding contact occurs and the rod's tip has a negative final tangential velocity; otherwise ($\mu > 0.6$), sticking contact occurs and the final tangential tip velocity is zero. These results agree with those in (Brach, 1989).

Case 2: Reversed Sliding ($C_o = -1.0$ and $S_o = 0.6$). If the initial tangential velocity is 0.6 (Fig. 8), then we will have reversed sliding for $\mu < 0.6$ and sticking for $\mu > 0.6$. Assuming the sticking contact, $\mu > 0.6$, with Newton's law of restitution, there is a net gain in energy. The impulses and resultant motions are

$$P_x = 0.375\ e$$

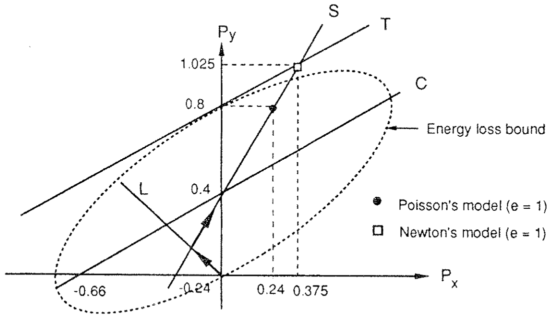

Fig. 8 Impact process diagram of the falling rod for case 2. Initial compression velocity $C_o = -1.0$ and initial sliding velocity $S_o = 0.6$. The ellipse denotes the boundary of the region of energy loss. The line of termination T is tangent to the ellipse.

$$P_y = 0.4 + 0.0625\, e$$
$$S = 0$$
$$C = e$$

and the energy gain of the system is

$$\Delta T = \frac{1}{2}\,(0.625\, e^2 - 0.4).$$

Therefore, for all values $e > 0.8$, the rod gains energy instead of losing energy. In order to lose energy, the final impulse must lie within the ellipse plotted in Fig. 8. For $e = 1$, the rod gains energy for any $\mu > 0$.

The difficulty does not occur if we use Poisson's hypothesis instead, giving

$$P_x = 0.24\, e$$
$$P_y = 0.4(1+e)$$
$$S = 0$$
$$C = 0.64\, e$$

with a corresponding energy increase

$$\Delta T = \frac{1}{2}\,(0.256\, e^2 - 0.4)$$

which is always negative for $0 \le e \le 1$. Poisson's model results in both kinematically and dynamically valid solutions.

Brach (1989) uses the same example, with Newton's law of restitution, and resolves the increase in energy in a very different manner. He does not treat μ (or e) as predetermined parameters. For this example, he disallows nonzero μ, because it would lead to energy gains.

Case 3: Forward Sliding; Sticking ($C_o = -1.0$ and $S_o = -1.0$). For initial conditions of $S_o = C_o = -1.0$, Fig. 9 shows the impact process diagram. The critical values of μ are $\mu_d = (3(1+e)+5)/(5(1+e)+3)$ and $\mu_q = 1.0$. If $\mu < \mu_d$, the tip keeps forward sliding in the collision; if $\mu_d < \mu < 1.0$, sticking in restitution occurs; and if $\mu > 1.0$, sticking in compression occurs.

Case 4: Tangential Impact ($C_o = 0$ and $S_o = -0.2$). The final example involves tangential impact, which was defined and discussed in Section 3.4. Initially the rod is sliding along the surface with zero normal velocity. We begin by considering a finite-force approach to the problem. We also modify the problem slightly: We introduce a gravitational field. The surprising result is that no solution exists: Every contact force consistent with Coulomb's law will violate the kinematic constraint. If the contact force were zero, the gravitational force would accelerate the tip downward. For positive contact forces, and with $\mu > 1.666$, the rod's physical parameters have been chosen so that the angular acceleration, accelerating the tip downward, dominates the linear acceleration, which would accelerate the tip upward. Mason and Wang (1988) and Wang (1989) present a more detailed analysis of the problem. Previous work, neglecting the possibility of an impact solution, present this example and variations to demonstrate the inconsistency of rigid-body mechanics (Lotstedt, 1981; Erdmann, 1984; Beghin, 1923–1924; Klein, 1909; Painleve, 1895; Hamel, 1949).

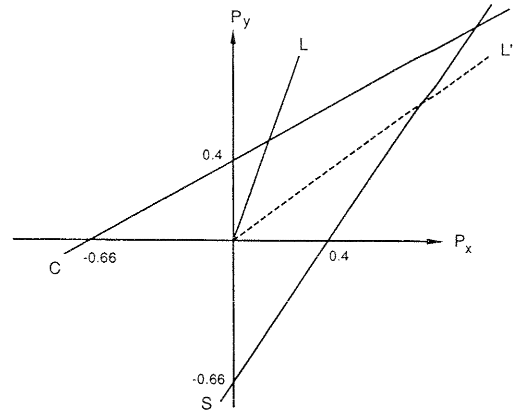

Fig. 9 Impact process diagram of the falling rod for case 3. Initial compression velocity $C_o = -1.0$ and initial sliding velocity $S_o = -1.0$. L and L' denote the lines of limiting friction for $\mu < \mu_d$ and $\mu > \mu_d$, respectively.

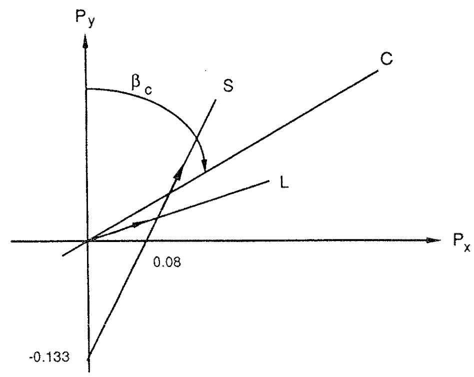

Fig. 10 Impact process diagram of the falling rod for case 4. Initial compression velocity $C_o = 0$ and initial sliding velocity $S_o = -0.2$

Now we apply the Routh-Poisson method to derive impulsive forces. In the impact process diagram (Fig. 10), the line of maximum compression C passes the origin with an angle $\beta_c = \tan^{-1} 1.666$. For $\mu > 1.666$, an impact solution with sticking contact exists, and both nonzero tangential and normal impulses are obtained. If $\mu < 1.666$, then compression cannot occur, so impulsive forces will be zero. An impact solution exists in exactly those cases where the finite solution does not exist.

How does Newton's law of restitution relate to tangential impact? Since the normal velocity is zero, the final velocity would also be zero, no matter what value for the coefficient of restitution. Hence, any difference between plastic and elastic behavior cannot be expressed using Newton's law.

7 Summary and Conclusion

This paper derives solutions for frictional planar rigid-body collisions, using Routh's impact process diagrams, for both Newtonian and Poisson restitution. We apply the graphical method of Routh to describe an analytical solution to the collision problem. The contact mode determines how the body velocities change during the course of impact. If the contact mode of impact is not properly identified, solutions sometimes violate energy conservation. Using Routh's method, we characterize all possible contact modes of impact, and then derive analytical solutions for impulses and motions of the bodies.

An important observation regards the definition of the coefficient of restitution. We have presented solutions using both Poisson's hypothesis and Newton's law of restitution. Poisson's hypothesis, relating the normal impulses during two postulated phases of compression and restitution, guarantees energy conservation principles, but Newton's law of restitution, relating the initial and final normal velocities, cannot. As a dynamic law, Poisson's hypothesis is superior to Newton's law of restitution which is an artificial kinematic relationship and is not always applicable.

There are alternative methods to resolve the violation of energy conservation. Rather than blaming the definition of restitution, it is possible to blame the definition of friction. Brach (1989) takes this approach, and adopts a coefficient of friction that is lowered to prevent energy gains, and also to prevent reversal of tangential tip velocity. In the most extreme cases, the only value of μ that satisfies these constraints is zero. We view Poisson restitution as preferable to a law that determines μ after the fact. In addition, Poisson's hypothesis works nicely with tangential impact. Stronge (1990) proposes an alternative law of restitution which also appears to resolve the energy conservation problem.

As Routh indicated, the extension of his approach to the general three-dimensional rigid-body impact problem is not by any means straightforward. Keller (1986) provides an analytical extension to three dimensions, and also gives a fundamental development of the method. However, the simplicity of Routh's graphical approach does not extend to three dimensions. No algebraic relationships in impulse space can be found to describe limiting friction. Differential descriptions are necessary, and an analytical solution would be difficult at best.

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We would like to thank Randy Brost, Ken Goldberg, and Michael Peshkin for their many useful suggestions. Special thanks to J. B. Keller for his constructive comments on an earlier report of the work. This work was supported under grants from the System Development Foundation and the National Science Foundation under grant DMC-8520475.

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APPENDIX

We verify energy conservation for Poisson's hypothesis for the perfect elastic case ($e = 1$). It is useful to note that $P_y \geq 0$, $B_1 > 0$, $B_2 > 0$, and $B_1B_2 - B_3^2 > 0$. The energy losses are given by contact mode:

- sliding:

$$2\Delta T = -s\mu P_y(S+S_o).$$

Note that S remains the same sign with S_o and $s(S + S_o) \geq 0$. Therefore, $\Delta T \leq 0$.

For the remaining contact modes, if we solve for S_o and C_o from the solutions of impulse given in Section 3.5 and substitute them in Eq. (57), then the energy change is a quadratic form of variables of impulses (P_x P_y) can be examined to determine its sign.

- C-sticking:

$$2\Delta T = -\frac{1}{B_1}\,(B_1P_x - B_3P_y)^2 \leq 0.$$

- R-sticking:

$$2\Delta T = -\,(B_1P_x^2 + s\mu B_3P_y^2).$$

There are two cases, $sB_3 \geq 0$ (therefore, $\Delta T \leq 0$) and $sB_3 < 0$. In the second case, the quadratic form is hyperbolic and $\Delta T \leq 0$ requires that $|P_x| \geq \sqrt{\mu|\mu_s|}P_y$, where $\mu_s = -B_3/B_1$ (Table 1). The conditions of the contact mode (Table 1) can be written as

$$-sP_x \geq \frac{1}{2}\,(\mu + |\mu_s|)P_y \text{ and } -sP_s \leq \mu P_y.$$

Since $\mu \geq |\mu_s|$ and $\mu \geq 1/2(\mu + |\mu_s|) \geq \sqrt{\mu|\mu_s|}$, it is evident that within these constraints the energy change $\Delta T \leq 0$ is true.

- C-reversed sliding:

$$2\Delta T = -\left[\frac{sB_3}{\mu}\,P_x^2 - (B_3 + s\mu B_1)P_xP_y + s\mu B_3P_y^2\right].$$

The conditions for the contact mode are $sB_3 > 0$ and $\mu \leq |\mu_s|$, where $\mu_s = -B_3/B_1$.

The determinant of the quadratic form is found to be

$$\Delta = B_3^2\left[1 - \frac{1}{4}\,(1 + \mu/|\mu_s|)^2\right] \geq 0.$$

Therefore, $\Delta T \leq 0$.

- R-reversed sliding:

$$2\Delta T = -\left[\frac{sB_3}{\mu}\,P_x^2 + (B_3 - s\mu B_1)P_xP_y + s\mu B_3P_y^2\right].$$

Considering the conditions $sB_3 > 0$ and $0 < \mu \leq |\mu_s|$, we found the determinant of the quadratic form to be

$$\Delta = B_3^2\left[1 - \frac{1}{4}\,(1 - \mu/|\mu_s|)^2\right] \geq 0$$

yielding $\Delta T \leq 0$.